A short HAP tutorial

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Contents

Chapter 1

Simplicial complexes & CW complexes

1.1 The Klein bottle as a simplicial complex

The following example constructs the Klein bottle as a simplicial complex K on 9 vertices, and then constructs the cellular chain complex $C_* = C_*(K)$ from which the integral homology groups $H_1(K,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}$, $H_2(K,\mathbb{Z}) = 0$ are computed. The chain complex $D_* = C_* \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is also constructed and used to compute the mod-2 homology vector spaces $H_1(K,\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H_2(K,\mathbb{Z}) = \mathbb{Z}_2$. Finally, a presentation $\pi_1(K) = \langle x, y : yxy^{-1}x \rangle$ is computed for the fundamental group of K.

```
――――――――――――――――― Example ―――――――――――――――――
gap> 2simplices:=
> [[1,2,5], [2,5,8], [2,3,8], [3,8,9], [1,3,9], [1,4,9],
>  [4,5,8], [4,6,8], [6,8,9], [6,7,9], [4,7,9], [4,5,7],
>  [1,4,6], [1,2,6], [2,6,7], [2,3,7], [3,5,7], [1,3,5]];;
gap> K:=SimplicialComplex(2simplices);
Simplicial complex of dimension 2.

gap> C:=ChainComplex(K);
Chain complex of length 2 in characteristic 0 .

gap> Homology(C,1);
[ 2, 0 ]
gap> Homology(C,2);
[  ]

gap> D:=TensorWithIntegersModP(C,2);
Chain complex of length 2 in characteristic 2 .

gap> Homology(D,1);
2
gap> Homology(D,2);
1

gap> G:=FundamentalGroup(K);
<fp group of size infinity on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(G);
[ f2*f1*f2^-1*f1 ]
```

1.2 Other simplicial surfaces

The following example constructs the real projective plane P, the Klein bottle K and the torus T as simplicial complexes, using the surface genus g as input in the oriented case and $-g$ as input in the unoriented cases. It then confirms that the connected sums $M = K\#P$ and $N = T\#P$ have the same integral homology.

```
─────────────────────── Example ───────────────────────
gap> P:=ClosedSurface(-1);
Simplicial complex of dimension 2.

gap> K:=ClosedSurface(-2);
Simplicial complex of dimension 2.

gap> T:=ClosedSurface(1);
Simplicial complex of dimension 2.

gap> M:=ConnectedSum(K,P);
Simplicial complex of dimension 2.

gap> N:=ConnectedSum(T,P);
Simplicial complex of dimension 2.

gap> Homology(M,0);
[ 0 ]
gap> Homology(N,0);
[ 0 ]
gap> Homology(M,1);
[ 2, 0, 0 ]
gap> Homology(N,1);
[ 2, 0, 0 ]
gap> Homology(M,2);
[  ]
gap> Homology(N,2);
[  ]
```

1.3 The Quillen complex

Given a group G one can consider the partially ordered set $\mathscr{A}_p(G)$ of all non-trivial elementary abelian p-subgroups of G, the partial order being set inclusion. The order complex $\Delta\mathscr{A}_p(G)$ is a simplicial complex which is called the *Quillen complex* .

The following example constructs the Quillen complex $\Delta\mathscr{A}_2(S_7)$ for the symmetric group of degree 7 and $p = 2$. This simplicial complex involves 11291 simplices, of which 4410 are 2-simplices..

```
─────────────────────── Example ───────────────────────
gap> K:=QuillenComplex(SymmetricGroup(7),2);
Simplicial complex of dimension 2.

gap> Size(K);
11291
```

```
gap> K!.nrSimplices(2);
4410
```

1.4   The Quillen complex as a reduced CW-complex

Any simplicial complex K can be regarded as a regular CW complex. Different datatypes are used in HAP for these two notions. The following continuation of the above Quillen complex example constructs a regular CW complex Y isomorphic to (i.e. with the same face lattice as) $K = \Delta\mathscr{A}_2(S_7)$. An advantage to working in the category of CW complexes is that it may be possible to find a CW complex X homotopy equivalent to Y but with fewer cells than Y. The cellular chain complex $C_*(X)$ of such a CW complex X is computed by the following commands. From the number of free generators of $C_*(X)$, which correspond to the cells of X, we see that there is a single 0-cell and 160 2-cells. Thus the Quillen complex $$\Delta{\cal A}_2(S_7) \simeq \bigvee_{1\le i\le 160} S^2$$ has the homotopy type of a wedge of 160 2-spheres. This homotopy equivalence is given in [Kso00, (15.1)] where it was obtained by purely theoretical methods.

```
──────── Example ────────
gap> Y:=RegularCWComplex(K);
Regular CW-complex of dimension 2

gap> C:=ChainComplex(Y);
Chain complex of length 2 in characteristic 0 .

gap> C!.dimension(0);
1
gap> C!.dimension(1);
0
gap> C!.dimension(2);
160
```

1.5   Simple homotopy equivalences

For any regular CW complex Y one can look for a sequence of simple homotopy collapses $Y \searrow Y_1 \searrow Y_2 \searrow \ldots \searrow Y_N = X$ with X a smaller, and typically non-regular, CW complex. Such a sequence of collapses can be recorded using what is now known as a *discrete vector field* on Y. The sequence can, for example, be used to produce a chain homotopy equivalence $f: C_*Y \to C_*X$ and its chain homotopy inverse $g: C_*X \to C_*Y$. The function ChainComplex(Y) returns the cellular chain complex $C_*(X)$, wheras the function ChainComplexOfRegularCWComplex(Y) returns the chain complex $C_*(Y)$.

For the above Quillen complex $Y = \Delta\mathscr{A}_2(S_7)$ the following commands produce the chain homotopy equivalence $f: C_*Y \to C_*X$ and $g: C_*X \to C_*Y$. The number of generators of C_*Y equals the number of cells of Y in each degree, and this number is listed for each degree.

```
──────── Example ────────
gap> K:=QuillenComplex(SymmetricGroup(7),2);;
gap> Y:=RegularCWComplex(K);;
gap> CY:=ChainComplexOfRegularCWComplex(Y);
Chain complex of length 2 in characteristic 0 .
```

```
gap> CX:=ChainComplex(Y);
Chain complex of length 2 in characteristic 0 .

gap> equiv:=ChainComplexEquivalenceOfRegularCWComplex(Y);;
gap> f:=equiv[1];
Chain Map between complexes of length 2 .

gap> g:=equiv[2];
Chain Map between complexes of length 2 .


gap> CY!.dimension(0);
1316
gap> CY!.dimension(1);
5565
gap> CY!.dimension(2);
4410
```

1.6   Cellular simplifications preserving homeomorphism type

For some purposes one might need to simplify the cell structure on a regular CW-complex *Y* so as to obtained a homeomorphic CW-complex *W* with fewer cells.

The following commands load a 4-dimensional simplicial complex *Y* representing the K3 complex surface. Its simplicial structure is taken from [SK11] and involves 1704 cells of various dimensions. The commands then convert the cell structure into that of a homeomorphic regular CW-complex *W* involving 774 cells.

```
───────────────────── Example ─────────────────────
gap> Y:=RegularCWComplex(SimplicialK3Surface());
Regular CW-complex of dimension 4

gap> Size(Y);
1704
gap> W:=SimplifiedComplex(Y);
Regular CW-complex of dimension 4

gap> Size(W);
774
```

1.7   Constructing a CW-structure on a knot complement

The following commands construct the complement $M = S^3 \setminus K$ of the trefoil knot *K*. This complement is returned as a 3-manifold *M* with regular CW-structure involving four 3-cells.

```
───────────────────── Example ─────────────────────
gap> arc:=ArcPresentation(PureCubicalKnot(3,1));
[ [ 2, 5 ], [ 1, 3 ], [ 2, 4 ], [ 3, 5 ], [ 1, 4 ] ]
gap> S:=SphericalKnotComplement(arc);
```

```
  Regular CW-complex of dimension 3

gap> S!.nrCells(3);
4
```

The following additional commands then show that *M* is homotopy equivalent to a reduced CW-complex *Y* of dimension 2 involving one 0-cell, two 1-cells and one 2-cell. The fundamental group of *Y* is computed and used to calculate the Alexander polynomial of the trefoil knot.

```
―――――――――――――――――――― Example ――――――――――――――――――――
gap> Y:=ContractedComplex(S);
Regular CW-complex of dimension 2

gap> CriticalCells(Y);
[ [ 2, 1 ], [ 1, 9 ], [ 1, 11 ], [ 0, 22 ] ]
gap> G:=FundamentalGroup(Y);;
gap> AlexanderPolynomial(G);
x_1^2-x_1+1
```

1.8 Constructing a regular CW-complex by attaching cells

The following example creates the projective plane *Y* as a regular CW-complex, and tests that it has the correct integral homology $H_0(Y,\mathbb{Z}) = \mathbb{Z}$, $H_1(Y,\mathbb{Z}) = \mathbb{Z}_2$, $H_2(Y,\mathbb{Z}) = 0$.

```
―――――――――――――――――――― Example ――――――――――――――――――――
gap> attch:=RegularCWComplex_AttachCellDestructive;; #Function for attaching cells

gap> Y:=RegularCWDiscreteSpace(3); #Discrete CW-complex consisting of points {1,2,3}
Regular CW-complex of dimension 0

gap> e1:=attch(Y,1,[1,2]);; #Attach 1-cell
gap> e2:=attch(Y,1,[1,2]);; #Attach 1-cell
gap> e3:=attch(Y,1,[1,3]);; #Attach 1-cell
gap> e4:=attch(Y,1,[1,3]);; #Attach 1-cell
gap> e5:=attch(Y,1,[2,3]);; #Attach 1-cell
gap> e6:=attch(Y,1,[2,3]);; #Attach 1-cell
gap> f1:=attch(Y,2,[e1,e3,e5]);; #Attach 2-cell
gap> f2:=attch(Y,2,[e2,e4,e5]);; #Attach 2-cell
gap> f3:=attch(Y,2,[e2,e3,e6]);; #Attach 2-cell
gap> f4:=attch(Y,2,[e1,e4,e6]);; #Attach 2-cell
gap> Homology(Y,0);
[ 0 ]
gap> Homology(Y,1);
[ 2 ]
gap> Homology(Y,2);
[  ]‘
```

The following example creates a 2-complex *K* corresponding to the group presentation
$G = \langle x,y,z \ : \ xyx^{-1}y^{-1} = 1, yzy^{-1}z^{-1} = 1, zxz^{-1}x^{-1} = 1 \rangle$.

The complex is shown to have the correct fundamental group and homology (since it is the 2-skeleton of the 3-torus $S^1 \times S^1 \times S^1$).

```
———————————————————————— Example ————————————————————————
gap> S1:=RegularCWSphere(1);;
gap> W:=WedgeSum(S1,S1,S1);;
gap> F:=FundamentalGroupWithPathReps(W);; x:=F.1;;y:=F.2;;z:=F.3;;
gap> K:=RegularCWComplexWithAttachedRelatorCells(W,F,Comm(x,y),Comm(y,z),Comm(x,z));
Regular CW-complex of dimension 2

gap> G:=FundamentalGroup(K);
<fp group on the generators [ f1, f2, f3 ]>
gap> RelatorsOfFpGroup(G);
[ f2^-1*f1*f2*f1^-1, f1^-1*f3*f1*f3^-1, f2^-1*f3*f2*f3^-1 ]
gap> Homology(K,1);
[ 0, 0, 0 ]
gap> Homology(K,2);
[ 0, 0, 0 ]
```

1.9 Constructing a regular CW-complex from its face lattice

The following example creats a 2-dimensional annulus A as a regular CW-complex, and testing that it has the correct integral homology $H_0(A,\mathbb{Z}) = \mathbb{Z}$, $H_1(A,\mathbb{Z}) = \mathbb{Z}$, $H_2(A,\mathbb{Z}) = 0$.

```
———————————————————————— Example ————————————————————————
gap> FL:=[];; #The face lattice
gap> FL[1]:=[[1,0],[1,0],[1,0],[1,0]];;
gap> FL[2]:=[[2,1,2],[2,3,4],[2,1,4],[2,2,3],[2,1,4],[2,2,3]];;
gap> FL[3]:=[[4,1,2,3,4],[4,1,2,5,6]];;
gap> FL[4]:=[];;
gap> A:=RegularCWComplex(FL);
Regular CW-complex of dimension 2

gap> Homology(A,0);
[ 0 ]
gap> Homology(A,1);
[ 0 ]
gap> Homology(A,2);
[  ]
```

Next we construct the direct product $Y = A \times A \times A \times A \times A$ of five copies of the annulus. This is a 10-dimensional CW complex involving 248832 cells. It will be homotopy equivalent $Y \simeq X$ to a CW complex X involving fewer cells. The CW complex X may be non-regular. We compute the cochain complex $D_* = \mathrm{Hom}_{\mathbb{Z}}(C_*(X), \mathbb{Z})$ from which the cohomology groups

$H^0(Y,\mathbb{Z}) = \mathbb{Z}$,
$H^1(Y,\mathbb{Z}) = \mathbb{Z}^5$,
$H^2(Y,\mathbb{Z}) = \mathbb{Z}^{10}$,
$H^3(Y,\mathbb{Z}) = \mathbb{Z}^{10}$,
$H^4(Y,\mathbb{Z}) = \mathbb{Z}^5$,

$H^5(Y, \mathbb{Z}) = \mathbb{Z}$,
$H^6(Y, \mathbb{Z}) = 0$
are obtained.

```
─────── Example ───────
gap> Y:=DirectProduct(A,A,A,A,A);
Regular CW-complex of dimension 10

gap> Size(Y);
248832
gap> C:=ChainComplex(Y);
Chain complex of length 10 in characteristic 0 .

gap> D:=HomToIntegers(C);
Cochain complex of length 10 in characteristic 0 .

gap> Cohomology(D,0);
[ 0 ]
gap> Cohomology(D,1);
[ 0, 0, 0, 0, 0 ]
gap> Cohomology(D,2);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Cohomology(D,3);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Cohomology(D,4);
[ 0, 0, 0, 0, 0 ]
gap> Cohomology(D,5);
[ 0 ]
gap> Cohomology(D,6);
[  ]
```

1.10 Cup products

STRATEGY 1: USE GEOMETRIC GROUP THEORY IN LOW DIMENSIONS.

Continuing with the previous example, we consider the first and fifth generators $g_1^1, g_5^1 \in H^1(Y, \mathbb{Z}) = \mathbb{Z}^5$ and establish that their cup product $g_1^1 \cup g_5^1 = -g_7^2 \in H^2(Y, \mathbb{Z}) = \mathbb{Z}^{10}$ is equal to minus the seventh generator of $H^2(Y, \mathbb{Z})$. We also verify that $g_5^1 \cup g_1^1 = -g_1^1 \cup g_5^1$.

```
─────── Example ───────
gap> cup11:=CupProduct(FundamentalGroup(Y));
function( a, b ) ... end

gap> cup11([1,0,0,0,0],[0,0,0,0,1]);
[ 0, 0, 0, 0, 0, 0, -1, 0, 0, 0 ]

gap> cup11([0,0,0,0,1],[1,0,0,0,0]);
[ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ]
```

This computation of low-dimensional cup products is achieved using group-theoretic methods to approximate the diagonal map $\Delta : Y \to Y \times Y$ in dimensions ≤ 2. In order to construct cup products in higher degrees HAP invokes three further strategies.

STRATEGY 2: IMPLEMENT THE ALEXANDER-WHITNEY MAP FOR A FIXED CLASS OF SPACES.

For simplicial complexes the cup product is implemented using the standard formula for the Alexander-Whitney chain map, together with homotopy equivalences to improve efficiency.

As a first example, the following commands construct simplicial surfaces $K = (\mathbb{S}^1 \times \mathbb{S}^1)\#(\mathbb{S}^1 \times \mathbb{S}^1)$ and $L = (\mathbb{S}^1 \times \mathbb{S}^1) \vee \mathbb{S}^1 \vee \mathbb{S}^1$ and establish that they have the same cohomology groups. It is then shown that the cup products $\cup_K : H^1(K,\mathbb{Z}) \times H^1(K,\mathbb{Z}) \to H^2(K,\mathbb{Z})$ and $\cup_L : H^1(L,\mathbb{Z}) \times H^1(L,\mathbb{Z}) \to H^2(L,\mathbb{Z})$ are antisymmetric bilinear forms of different ranks; hence K and L have different homotopy types.

—————————————— Example ——————————————
```
gap> K:=ClosedSurface(2);
Simplicial complex of dimension 2.

gap> L:=WedgeSum(WedgeSum(ClosedSurface(1),Sphere(1)),Sphere(1));
Simplicial complex of dimension 2.

gap> Cohomology(K,0);Cohomology(L,0);
[ 0 ]
[ 0 ]
gap> Cohomology(K,1);Cohomology(L,1);
[ 0, 0, 0, 0 ]
[ 0, 0, 0, 0 ]
gap> Cohomology(K,2);Cohomology(L,2);
[ 0 ]
[ 0 ]
gap> gens:=[[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]];;
gap> cupK:=CupProduct(K);;
gap> cupL:=CupProduct(L);;
gap> A:=NullMat(4,4);;B:=NullMat(4,4);;
gap> for i in [1..4] do
> for j in [1..4] do
> A[i][j]:=cupK(1,1,gens[i],gens[j])[1];
> B[i][j]:=cupL(1,1,gens[i],gens[j])[1];
> od;od;
gap> Display(A);
[ [   0,   0,   0,   1 ],
  [   0,   0,   1,   0 ],
  [   0,  -1,   0,   0 ],
  [  -1,   0,   0,   0 ] ]
gap> Display(B);
[ [   0,   1,   0,   0 ],
  [  -1,   0,   0,   0 ],
  [   0,   0,   0,   0 ],
  [   0,   0,   0,   0 ] ]
gap> Rank(A);
4
gap> Rank(B);
2
```

As a second example of the computation of cups products, the following commands construct the

wedge sum $W = M \vee N$ of a 2-dimensional orientable simplicial surface of genus 2 and the $K3$ complex surface which is stored as a pure simplicial complex of dimension 4. The simplicial structure on the $K3$ surface is taken from [SK11]. The commands then show that $H^2(W, \mathbb{Z}) = \mathbb{Z}^{22}$ and $H^4(W, \mathbb{Z}) = \mathbb{Z}$. The final commands comput the matrix $(x \cup y)$ as x, y range over a generating set for $H^2(W, \mathbb{Z})$.

```
————————————————————————— Example ——————————————————————————
gap> M:=ClosedSurface(2);;
gap> N:=SimplicialK3Surface();;
gap> W:=WedgeSum(M,N);
Simplicial complex of dimension 4.

gap> Cohomology(W,0);
[ 0 ]
gap> Cohomology(W,1);
[ 0, 0, 0, 0 ]
gap> Cohomology(W,2);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Cohomology(W,3);
[  ]
gap> Cohomology(W,4);
[ 0 ]
gap> cup:=CupProduct(W);;
gap> SecondCohomologyGens:=IdentityMat(23);;
gap> A:=NullMat(23,23);;
gap> for i in [1..23] do
> for j in [1..23] do
> A[i][j]:=cup(2,2,SecondCohomologyGens[i],SecondCohomologyGens[j])[1];
> od;od;
gap> Display(A);
[ [   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0,   0
  [   0,  -2,   2,  -1,   0,  -1,   1,  -1,   0,   0,   1,  -1,   0,   0,   1,   1,   2,  -1,  -1
  [   0,   2,  -4,   2,  -1,   1,  -1,   1,   0,  -1,  -1,   0,   1,  -1,   1,  -1,  -2,   1,   2
  [   0,  -1,   2,  -2,   2,  -1,   0,  -1,   1,   1,   1,   0,   0,   0,  -2,   0,   0,   0,   0
  [   0,   0,  -1,   2,  -4,   1,   1,   1,  -2,  -1,  -1,   0,  -1,  -1,   4,   0,   1,  -1,  -1
  [   0,  -1,   1,  -1,   1,  -2,   0,  -1,   1,   0,   1,  -1,   1,   0,   0,   1,   1,   0,   0
  [   0,   1,  -1,   0,   1,   0,  -2,   1,   0,   0,  -1,   0,   1,   1,  -1,   0,  -1,   0,   0
  [   0,  -1,   1,  -1,   1,  -1,   1,  -2,   1,   0,   1,  -1,   1,   0,   0,   1,   1,   0,   0
  [   0,   0,   0,   1,  -2,   1,   0,   1,  -2,  -1,   0,   0,  -1,  -1,   2,   0,   0,  -1,   0
  [   0,   0,  -1,   1,  -1,   0,   0,   0,  -1,  -2,   0,  -1,   0,   0,   2,   1,   0,   0,   0
  [   0,   1,  -1,   1,  -1,   1,  -1,   1,   0,   0,  -2,   0,   1,   1,  -1,   0,  -1,   0,   0
  [   0,  -1,   0,   0,   0,  -1,   0,  -1,   0,  -1,   0,  -2,   1,   0,   2,   1,   1,  -1,   0
  [   0,   0,   1,   0,  -1,   1,   1,   1,  -1,   0,   1,   1,  -2,   0,   0,   0,   0,   1,  -1
  [   0,   0,  -1,   0,  -1,   0,   1,   0,  -1,   0,   1,   0,   0,  -2,   3,   0,   1,  -1,   0
  [   0,   1,   1,  -2,   4,   0,  -1,   0,   2,   2,  -1,   2,   0,   3,  -8,  -1,  -2,   1,   1
  [   0,   1,  -1,   0,   0,   1,   0,   1,   0,   1,   0,   1,   0,   0,  -1,  -2,  -1,   1,   1
  [   0,   2,  -2,   0,   1,   1,  -1,   1,   0,   0,  -1,   1,   0,   1,  -2,  -1,  -2,   0,   2
  [   0,  -1,   1,   0,  -1,   0,   0,   0,  -1,   0,   0,  -1,   1,  -1,   1,   1,   0,  -2,  -1
  [   0,  -1,   2,   0,  -1,   0,   0,   0,   0,   0,   0,   0,  -1,   0,   1,   1,   2,  -1,  -2
  [   0,   1,  -2,   1,  -1,   1,   0,   1,  -1,   0,   0,   0,   1,  -1,   1,  -1,  -1,   0,   1
  [   0,  -1,   1,   0,  -1,   0,   1,   0,  -1,   0,   1,   0,  -1,  -1,   2,   0,   1,   0,  -1
  [   0,   0,   0,   1,  -1,   1,   0,   0,  -1,  -1,  -1,   0,  -1,   0,   0,   0,   0,  -1,   0
  [   0,  -1,   3,  -1,   1,   0,   0,   0,   0,   1,   0,   1,  -2,   1,  -2,   0,   0,   1,  -2
```

A cubical Alexander-Whitney formula, due to J.-P. Serre, could be used for computing the cohomology ring of a regular CW-complex whose cells all have a cubical combinatorial face lattice. This has not yet been implemented in HAP. However, the following more general approach has been implemented.

STRATEGY 3: IMPLEMENT A CELLULAR APPROXIMATION TO THE DIAGONAL MAP ON AN ARBITRARY FINITE REGULAR CW-COMPLEX.

The following example re-calculates the cup product on the wedge sum $W = M \vee N$ of a 2-dimensional orientable simplicial surface of genus 2 and the $K3$ complex surface which, this time, is stored as a finite regular CW-complex. Discrete vector field methods are used to construct a diagonal approximation. The simplicial structure of W is not used – in fact, the command W:=SimplifiedComplex(W) converts W to a regular CW-complex whose cells are typically not simplices. The method uses a different basis for the cohomology to that used above and thus produces a different 'cup product matrix".

```
───────────────────────── Example ─────────────────────────
gap> M:=ClosedSurface(2);;
gap> N:=SimplicialK3Surface();;
gap> W:=WedgeSum(M,N);;
gap> W:=RegularCWComplex(W);
Regular CW-complex of dimension 4

gap> W:=SimplifiedComplex(W);
Regular CW-complex of dimension 4

gap> cup:=CupProduct(W);;
gap> SecondCohomologyGens:=IdentityMat(23);;
gap> A:=NullMat(23,23);;
gap> for i in [1..23] do
> for j in [1..23] do
> A[i][j]:=cup(2,2,SecondCohomologyGens[i],SecondCohomologyGens[j])[1];
> od;od;
gap> Display(A);
[ [   -3,    1,    1,    0,    0,   -4,    1,    1,   -1,    2,    0,   -1,   -1,   -6,    0,
  [    0,   -1,   -3,    2,   -2,    5,    0,    0,    0,   -1,    1,    0,    1,    6,    0,
  [    0,   -1,   -2,    3,   -3,  -10,    0,    2,   -1,    1,    1,   -1,    2,    6,    1,
  [    2,    1,    2,  -10,    4,    7,    0,   -3,    0,   -2,   -1,    2,    0,    5,   -2,
  [    1,   -1,   -2,    4,   -2,   -4,    0,    2,    0,    0,    4,   -3,    2,   -9,    2,
  [   -1,   -4,   -6,    0,   -6,   -7,   -5,    7,   -9,   -2,   -7,    8,   -3,   54,   -8,
  [    1,    0,   -1,    0,   -1,    0,   -2,    1,    0,   -2,   -2,    2,    0,    7,   -2,
  [    0,    0,    0,   -2,    2,    3,    2,   -3,    1,    0,    0,    0,    0,    3,    1,
  [    0,   -1,    0,   -2,   -2,   -4,   -3,    4,   -5,    1,   -2,    3,   -1,    9,   -4,
  [    1,    0,   -1,    0,    0,   -1,   -1,   -2,    2,   -2,   -1,    1,    0,    3,    0,
  [    0,    1,    2,   -1,   -1,  -16,   -3,    5,   -5,   -1,  -14,   11,   -3,   27,   -9,
  [   -1,    2,    0,    0,    3,   21,    3,   -5,    7,    1,   11,  -10,    2,  -30,    7,
  [    0,    1,    2,    0,    0,   -5,    1,    0,    0,    0,    0,   -1,    0,   -8,    0,
  [   -9,    3,    2,    3,   10,   58,    3,   -7,    6,    1,   -2,    3,  -11,  -22,    3,
  [    1,   -1,    1,   -1,   -2,  -11,   -1,    4,   -5,    0,   -3,    2,    1,   14,   -4,
  [    1,   -1,   -1,    1,   -2,   -8,   -2,    3,   -3,    0,   -2,    3,    1,    7,   -2,
  [   -2,    1,   -1,    3,    0,    7,    1,   -1,    1,    1,    2,   -5,    0,   -8,    2,
  [    1,   -1,   -1,    2,   -2,   -4,   -2,    3,   -2,    0,   -1,    3,    0,    6,   -2,
  [   -1,    1,    1,    0,    2,    5,    0,   -1,    1,    0,    0,   -1,   -1,   -6,    1,
  [    0,    1,   -1,    4,    1,   10,    1,   -2,    6,    0,    9,   -7,    2,  -29,    7,
  [    1,   -1,   -2,    2,   -4,  -12,   -2,    4,   -2,    0,   -4,    4,    0,   20,   -4,
```

```
[   -1,    1,    2,   -3,    2,    6,    2,   -2,    1,    2,    4,   -3,    0, | -13,    2,
[    2,   -2,   -1,    0,   -3,  -11,   -1,    2,   -2,    0,    0,    0,    2, |  10,   -1,   -
```

STRATEGY 4: GUESS AND VERIFY A CELLULAR APPROXIMATION TO THE DIAGONAL MAP.

Many naturally occuring cell structures are neither simplicial nor cubical. For a general regular CW-complex we can attempt to construct a cellular inclusion $\overline{Y} \hookrightarrow Y \times Y$ with $\{(y,y) \ : \ y \in Y\} \subset \overline{Y}$ and with projection $p: \overline{Y} \twoheadrightarrow Y$ that induces isomorphisms on integral homology. The function `DiagonalApproximation(Y)` constructs a candidate inclusion, but the projection $p: \overline{Y} \twoheadrightarrow Y$ needs to be tested for homology equivalence. If the candidate inclusion passes this test then the function `CupProductOfRegularCWComplex_alt(Y)`, involving the candidate space, can be used for cup products. (I think the test is passed for all regular CW-complexes that are subcomplexes of some Euclidean space with all cells convex polytopes – but a proof needs to be written down!)

The following example calculates $g_1^2 \cup g_2^2 \neq 0$ where $Y = T \times T$ is the direct product of two copies of a simplicial torus T, and where g_k^n denotes the k-th generator in some basis of $H^n(Y, \mathbb{Z})$. The direct product Y is a CW-complex which is not a simplicial complex.

```
———————————— Example ————————————
gap> K:=RegularCWComplex(ClosedSurface(1));;
gap> Y:=DirectProduct(K,K);;
gap> cup:=CupProductOfRegularCWComplex_alt(Y);;
gap> cup(2,2,[1,0,0,0,0,0],[0,1,0,0,0,0]);
[ 5 ]

gap> D:=DiagonalApproximation(Y);;
gap> p:=D!.projection;
Map of regular CW-complexes

gap> P:=ChainMap(p);
Chain Map between complexes of length 4 .

gap> IsIsomorphismOfAbelianFpGroups(Homology(P,0));
true
gap> IsIsomorphismOfAbelianFpGroups(Homology(P,2));
true
gap> IsIsomorphismOfAbelianFpGroups(Homology(P,3));
true
gap> IsIsomorphismOfAbelianFpGroups(Homology(P,4));
true
```

Of course, either of Strategies 2 or 3 could also be used for this example. To use the Alexander-Whitney formula of Strategy 2 we would need to give the direct product $Y = T \times T$ a simplicial structure. This could be obtained using the function `SCCartesianProduct(T,T)` from the SIMP-COMP package. The details are as follows. (The result is consistent with the preceding computation since the choice of a basis for cohomology groups is far from unique.)

```
———————————— Example ————————————
gap> K:=ClosedSurface(1);;
gap> facets:=MaximalSimplicesOfSimplicialComplex(K);;
gap> LoadPackage("simpcomp");;
gap> scK:=SCFromFacets(S);;
```

```
gap> scY:=SCCartesianProduct(scK,scK);;
gap> Y:=MaximalSimplicesToSimplicialComplex(SCFacetsEx(scY));
Simplicial complex of dimension 4.

gap> cup:=CupProduct(Y);;
gap> cup(2,2,[1,0,0,0,0,0],[0,1,0,0,0,0]);
[ 0 ]
```

1.11 Intersection forms of 4-manifolds

The cup product gives rise to the intersection form of a connected, closed, orientable 4-manifold Y is a symmetric bilinear form

$qY:H^2(Y,\mathbb{Z})/Torsion \times H^2(Y,\mathbb{Z})/Torsion \longrightarrow \mathbb{Z}$

which we represent as a symmetric matrix.

The following example constructs the direct product $L = S^2 \times S^2$ of two 2-spheres, the connected sum $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ of the complex projective plane $\mathbb{C}P^2$ and its oppositely oriented version $\overline{\mathbb{C}P^2}$, and the connected sum $N = \mathbb{C}P^2 \# \mathbb{C}P^2$. The manifolds L, M and N are each shown to have a CW-structure involving one 0-cell, two 1-cells and one 2-cell. They are thus simply connected and have identical cohomology.

```
                           ──────── Example ────────
  gap> S:=Sphere(2);;
  gap> S:=RegularCWComplex(S);;
  gap> L:=DirectProduct(S,S);
  Regular CW-complex of dimension 4

  gap> M:=ConnectedSum(ComplexProjectiveSpace(2),ComplexProjectiveSpace(2),-1);
  Simplicial complex of dimension 4.

  gap> N:=ConnectedSum(ComplexProjectiveSpace(2),ComplexProjectiveSpace(2),+1);
  Simplicial complex of dimension 4.

  gap> CriticalCells(L);
  [ [ 4, 1 ], [ 2, 13 ], [ 2, 56 ], [ 0, 16 ] ]
  gap> CriticalCells(RegularCWComplex(M));
  [ [ 4, 1 ], [ 2, 109 ], [ 2, 119 ], [ 0, 8 ] ]
  gap> CriticalCells(RegularCWComplex(N));
  [ [ 4, 1 ], [ 2, 119 ], [ 2, 149 ], [ 0, 12 ] ]
```

John Milnor showed (as a corollary to a theorem of J. H. C. Whitehead) that the homotopy type of a simply connected 4-manifold is determined by its quadratic form. More precisely, a form is said to be of *type I (properly primitive)* if some diagonal entry of its matrix is odd. If every diagonal entry is even, then the form is of *type II (improperly primitive)*. The *index* of a form is defined as the number of positive diagonal entries minus the number of negative ones, after the matrix has been diagonalized over the real numbers.

THEOREM. (Milnor [Mil58]) The oriented homotopy type of a simply connected, closed, orientable 4-manifold is determined by its second Betti number and the index and type of its intersetion form; except possibly in the case of a manifold with definite quadratic form of rank r > 9.

The following commands compute matrices representing the intersection forms qL, qM, qN.

```
────────────────────────── Example ──────────────────────────
gap> qL:=IntersectionForm(L);;
gap> qM:=IntersectionForm(M);;
gap> qN:=IntersectionForm(N);;
gap> Display(qL);
[ [  -2,   1 ],
  [   1,   0 ] ]
gap> Display(qM);
[ [  1,  0 ],
  [  0,  1 ] ]
gap> Display(qN);
[ [   1,   0 ],
  [   0,  -1 ] ]
```

Since qL is of type II, whereas qM and qN are of type I we see that the oriented homotopy type of L is distinct to that of M and that of N. Since qM has index 2 and qN has index 0 we see that that M and N also have distinct oriented homotopy types.

1.12   Cohomology Rings

The cup product gives the cohomology $H^*(X,R)$ of a space X with coefficients in a ring R the structure of a graded commutitive ring. The function CohomologyRing(Y,p) returns the cohomology as an algebra for Y a simplicial complex and $R = \mathbb{Z}_p$ the field of p elements. For more general regular CW-complexes or $R = \mathbb{Z}$ the cohomology ring structure can be determined using the function CupProduct(Y).

The folowing commands compute the mod 2 cohomology ring $H^*(W, \mathbb{Z}_2)$ of the above wedge sum $W = M \vee N$ of a 2-dimensional orientable simplicial surface of genus 2 and the $K3$ complex simplicial surface (of real dimension 4).

```
────────────────────────── Example ──────────────────────────
gap> M:=ClosedSurface(2);;
gap> N:=SimplicialK3Surface();;
gap> W:=WedgeSum(M,N);;
gap> A:=CohomologyRing(W,2);
<algebra of dimension 29 over GF(2)>
gap> x:=Basis(A)[25];
v.25
gap> y:=Basis(A)[27];
v.27
gap> x*y;
v.29
```

The functions CupProduct and IntersectionForm can be used to determine integral cohomology rings. For example, the integral cohomology ring of an arbitrary closed surface was calculated in [GM15, Theorem 3.5]. For any given surface M this result can be recalculated using the intersection form. For instance, for an orientable surface of genus g it is well-known that $H^1(M,\mathbb{Z}) = \mathbb{Z}^{2g}$, $H^2(M,\mathbb{Z}) = \mathbb{Z}$. The ring structure multiplication is thus given by the matrix of the intersection form. For say $g = 3$ the ring multiplication is given, with respect to some cohomology basis, in the following.

```
——————————————————— Example ———————————————————
  gap> M:=ClosedSurface(3);;
  gap> Display(IntersectionForm(M));
  [ [    0,    0,    1,   -1,   -1,    0 ],
    [    0,    0,    0,    1,    1,    0 ],
    [   -1,    0,    0,    1,    1,   -1 ],
    [    1,   -1,   -1,    0,    0,    1 ],
    [    1,   -1,   -1,    0,    0,    0 ],
    [    0,    0,    1,   -1,    0,    0 ] ]
```

By changing the basis B for $H^1(M,\mathbb{Z})$ we obtain the following simpler matrix representing multiplication in $H^*(M,\mathbb{Z})$.

```
——————————————————— Example ———————————————————
  gap> B:=[ [ 0, 1, -1, -1, 1, 0 ],
  >         [ 1, 0, 1, 1, 0, 0 ],
  >         [ 0, 0, 1, 0, 0, 0 ],
  >         [ 0, 0, 0, 1, -1, 0 ],
  >         [ 0, 0, 1, 1, 0, 0 ],
  >         [ 0, 0, 1, 1, 0, 1 ] ];;
  gap> Display(IntersectionForm(M,B));
  [ [    0,    1,    0,    0,    0,    0 ],
    [   -1,    0,    0,    0,    0,    0 ],
    [    0,    0,    0,    0,    1,    0 ],
    [    0,    0,    0,    0,    0,    1 ],
    [    0,    0,   -1,    0,    0,    0 ],
    [    0,    0,    0,   -1,    0,    0 ] ]
```

1.13   CW maps and induced homomorphisms

A *strictly cellular* map $f\colon X \to Y$ of regular CW-complexes is a cellular map for which the image of any cell is a cell (of possibly lower dimension). Inclusions of CW-subcomplexes, and projections from a direct product to a factor, are examples of such maps. Strictly cellular maps can be represented in HAP, and their induced homomorphisms on (co)homology and on fundamental groups can be computed.

The following example begins by visualizing the trefoil knot $\kappa \in \mathbb{R}^3$. It then constructs a regular CW structure on the complement $Y = D^3 \setminus \text{Nbhd}(\kappa)$ of a small tubular open neighbourhood of the knot lying inside a large closed ball D^3. The boundary of this tubular neighbourhood is a 2-dimensional CW-complex B homeomorphic to a torus $\mathbb{S}^1 \times \mathbb{S}^1$ with fundamental group $\pi_1(B) =< a,b : aba^{-1}b^{-1} = 1 >$. The inclusion map $f\colon B \hookrightarrow Y$ is constructed. Then a presentation $\pi_1(Y) =< x,y \,|\, xy^{-1}x^{-1}yx^{-1}y^{-1} >$ and the induced homomorphism $$\pi_1(B)\rightarrow \pi_1(Y), a\mapsto y^{-1}xy^2xy^{-1}, b\mapsto y $$ are computed. This induced homomorphism is an example of a *peripheral system* and is known to contain sufficient information to characterize the knot up to ambient isotopy.

Finally, it is verified that the induced homology homomorphism $H_2(B,\mathbb{Z}) \to H_2(Y,\mathbb{Z})$ is an isomomorphism.

---------- Example ----------

```
gap> K:=PureCubicalKnot(3,1);;
gap> ViewPureCubicalKnot(K);;
```

---------- Example ----------

```
gap> K:=PureCubicalKnot(3,1);;
gap> f:=KnotComplementWithBoundary(ArcPresentation(K));
Map of regular CW-complexes

gap> G:=FundamentalGroup(Target(f));
<fp group of size infinity on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(G);
[ f1*f2^-1*f1^-1*f2*f1^-1*f2^-1 ]

gap> F:=FundamentalGroup(f);
[ f1, f2 ] -> [ f2^-1*f1*f2^2*f1*f2^-1, f1 ]


gap> phi:=ChainMap(f);
Chain Map between complexes of length 2 .

gap> H:=Homology(phi,2);
[ g1 ] -> [ g1 ]
```

1.14 Constructing a simplicial complex from a regular CW-complex

The following example constructs a 3-dimensional pure regular CW-complex *K* whose 3-cells are permutahedra. It then constructs the simplicial complex *B* by taking barycentric subdivision. It then constructes a smaller, homotopy equivalent, simplicial complex *N* by taking the nerve of the cover of *K* by the closures of its 3-cells.

---------- Example ----------

```
gap> K:=RegularCWComplex(PureComplexComplement(PurePermutahedralKnot(3,1)));
Regular CW-complex of dimension 3

gap> Size(K);
77923
gap> B:=BarycentricSubdivision(K);
Simplicial complex of dimension 3.

gap> Size(B);
1622517
gap> N:=Nerve(K);
Simplicial complex of dimension 3.

gap> Size(N);
48745
```

1.15 Equivariant CW complexes

The above representations of spaces as simplicial complexes and regular CW complexes have their limitations. One limitation is that the number of cells needed to describe a space can be unnecessarily large. A minimal simplicial complex structure for the torus has 7 vertices, 21 edges and 14 triangles. A minimal regular CW-complex structure for the torus has 4 vertices, 8 edges and 4 cells of dimension 2. By using simplicial sets (which are like simplicial complexes except that they allow the freedom to attach simplicial cells by gluing their boundary non-homeomorphically) one obtains a minimal triangulation of the torus involving 1 vertex, 3 edges and 2 cells of dimension 2. By using non-regular CW-complexes one obtains a minimal cell structure involving 1 vertex, 2 edges and 1 cell of dimension 2. Minimal cell structures (in the four different categories) for the torus are illustrated as follows.

A second limitation to our representations of simplicial and regular CW-complexes is that they apply only to structures with finitely many cells. They do no apply, for instance, to the simplicial complex structure on the real line $\mathbb{R}$ in which each each integer n is a vertex and each interval $[n, n+1]$ is an edge.

Simplicial sets provide one approach to the efficient combinatorial representation of certain spaces. So too do cubical sets (the analogues of simplicial sets in which each cell has the combinatorics of an n-cube rather than an n-simplex). Neither of these two approaches has been implemented in HAP.

Simplicial sets endowed with the action of a (possibly infinite) group G provide for an efficient representation of (possibly infinite) cell structures on a wider class of spaces. Such a structure can be made precise and is known as a *simplicial group*. Some functionality for simplicial groups is implemented in HAP and described in Chapter 12.

A regular CW-complex endowed with the action of a (possibly infinite) group G is an alternative approach to the efficient combinatorial representation of (possibly infinite) cell structures on spaces. Much of HAP is focused on this approach. As a first example of the idea, the following commands construct the infinite regular CW-complex $Y = \widetilde{T}$ arising as the universal cover of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ where T is given the above minimal non-regular CW structure involving 1 vertex, 2 edges, and 1 cell of dimension 2. The homology $H_n(T, \mathbb{Z})$ is computed and the fundamental group of the torus T is recovered.

```
───────────────────────────── Example ─────────────────────────────
gap> F:=FreeGroup(2);;x:=F.1;;y:=F.2;;
gap> G:=F/[ x*y*x^-1*y^-1 ];;
gap> Y:=EquivariantTwoComplex(G);
Equivariant CW-complex of dimension 2

gap> C:=ChainComplexOfQuotient(Y);
Chain complex of length 2 in characteristic 0 .

gap> Homology(C,0);
[ 0 ]
gap> Homology(C,1);
[ 0, 0 ]
gap> Homology(C,2);
[ 0 ]
gap> FundamentalGroupOfQuotient(Y);
<fp group of size infinity on the generators [ f1, f2 ]>
```

As a second example, the following comands load group number 9 in the library of 3-dimensional

crystallographic groups. They verify that G acts freely on $\mathbb{R}^3$ (i.e. G is a *Bieberbach group*) and then construct a G-equivariant CW-complex $Y = \mathbb{R}^3$ corresponding to the tessellation of $\mathbb{R}^3$ by a fundamental domain for G. Finally, the cohomology $H_n(M, \mathbb{Z})$ of the 3-dimensional closed manifold $M = \mathbb{R}^3/G$ is computed. The manifold M is seen to be non-orientable (since it's top-dimensional homology is trivial) and has a non-regular CW structure with 1 vertex, 3 edges, 3 cells of dimension 2, and 1 cell of dimension 3. (This example uses Polymake software.)

```
                          ———————————— Example ————————————
gap> G:=SpaceGroup(3,9);;
gap> IsAlmostBieberbachGroup(Image(IsomorphismPcpGroup(G)));
true
gap> Y:=EquivariantEuclideanSpace(G,[0,0,0]);
Equivariant CW-complex of dimension 3

gap> Y!.dimension(0);
1
gap> Y!.dimension(1);
3
gap> Y!.dimension(2);
3
gap> Y!.dimension(3);
1
gap> C:=ChainComplexOfQuotient(Y);
Chain complex of length 3 in characteristic 0 .

gap> Homology(C,0);
[ 0 ]
gap> Homology(C,1);
[ 0, 0 ]
gap> Homology(C,2);
[ 2, 0 ]
gap> Homology(C,3);
[ ]
```

1.16 Orbifolds and classifying spaces

If a discrete group G acts on Euclidean space or hyperbolic space with finite stabilizer groups then we say that the quotient space obtained by killing the action of G an an *orbifold*. If the stabilizer groups are all trivial then the quotient is of course a manifold.

An orbifold is represented as a G-equivariant regular CW-complex together with the stabilizer group for a representative of each orbit of cells and its subgroup consisting of those group elements that preserve the cell orientation. HAP stores orbifolds using the data type of *non-free resolution* and uses them mainly as a first step in constructing free $\mathbb{Z}G$-resolutions of $\mathbb{Z}$.

The following commands use an 8-dimensional equivariant deformation retract of a $GL_3(\mathbb{Z}[\mathbf{i}])$-orbifold structure on hyperbolic space to compute $H_5(GL_3(\mathbb{Z}[\mathbf{i}]), \mathbb{Z}) = \mathbb{Z}_2^5 \oplus \mathbb{Z}_4^2$. (The deformation retract is stored in a library and was supplied by Mathieu Dutour Sikiric.)

```
                          ———————————— Example ————————————
gap> Orbifold:=ContractibleGcomplex("PGL(3,Z[i])");
Non-free resolution in characteristic 0 for matrix group .
No contracting homotopy available.
```

```
gap> R:=FreeGResolution(Orbifold,6);
Resolution of length 5 in characteristic 0 for matrix group .
No contracting homotopy available.

gap> Homology(TensorWithIntegers(R),5);
[ 2, 2, 2, 2, 2, 4, 4 ]
```

The next example computes an orbifold structure on $\mathbb{R}^4$, and then the first 12 degrees of a free resolution/classifying space, for the second 4-dimensional crystallographic group G in the library of crystallographic groups. The resolution is shown to be periodic of period 2 in degrees ≥ 5. The cohomology is seen to have 11 ring generators in degree 2 and no further ring generators. The cohomology groups are: $$H^n(G,\mathbb Z) =\left(\begin{array}{ll} 0, & {\rm odd~} n\ge 1\\ \mathbb Z_2^5 \oplus \mathbb Z^6, & n=2\\ \mathbb Z_2^{15}\oplus \mathbb Z, & n=4\\ \mathbb Z_2^{16}, & {\rm even~} n \ge 6 .\\ \end{array}\right.$$

```
                              ──────── Example ────────
gap> G:=SpaceGroup(4,2);;
gap> R:=ResolutionCubicalCrystGroup(G,12);
Resolution of length 12 in characteristic 0 for <matrix group with
5 generators> .

gap> R!.dimension(5);
16
gap> R!.dimension(7);
16
gap> List([1..16],k->R!.boundary(5,k)=R!.boundary(7,k));
[ true, true, true, true, true, true, true, true, true, true, true, true,
  true, true, true, true ]

gap> C:=HomToIntegers(R);
Cochain complex of length 12 in characteristic 0 .

gap> Cohomology(C,0);
[ 0 ]
gap> Cohomology(C,1);
[  ]
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0 ]
gap> Cohomology(C,3);
[  ]
gap> Cohomology(C,4);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0 ]
gap> Cohomology(C,5);
[  ]
gap> Cohomology(C,6);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
gap> Cohomology(C,7);
[  ]

gap> IntegralRingGenerators(R,1);
[  ]
```

```
gap> IntegralRingGenerators(R,2);
[ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ],
  [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ] ]
gap> IntegralRingGenerators(R,3);
[  ]
gap> IntegralRingGenerators(R,4);
[  ]
gap> IntegralRingGenerators(R,5);
[  ]
gap> IntegralRingGenerators(R,6);
[  ]
gap> IntegralRingGenerators(R,7);
[  ]
gap> IntegralRingGenerators(R,8);
[  ]
gap> IntegralRingGenerators(R,9);
[  ]
gap> IntegralRingGenerators(R,10);
[  ]
```

A group G with a finite index torsion free nilpotent subgroup admits a resolution which is periodic in sufficiently high degrees if and only if all of its finite index subgroups admit periodic resolutions. The following commands identify the 99 3-dimensional space groups (respectively, the 1191 4-dimensional space groups) that admit a resolution which is periodic in degrees > 3 (respectively, in degrees > 4).

```
——————————————— Example ———————————————
gap> L3:=Filtered([1..219],k->IsPeriodicSpaceGroup(SpaceGroup(3,k)));
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 17, 18, 19, 21, 24, 26, 27, 28,
  29, 30, 31, 32, 33, 34, 36, 37, 39, 40, 41, 43, 45, 46, 52, 54, 55, 56, 58,
  61, 62, 74, 75, 76, 77, 78, 79, 80, 81, 84, 85, 87, 89, 92, 98, 101, 102,
  107, 111, 119, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151,
  152, 153, 154, 155, 157, 159, 161, 162, 163, 164, 165, 166, 168, 171, 172,
  174, 175, 176, 178, 180, 186, 189, 192, 196, 198, 209 ]

gap> L4:=Filtered([1..4783],k->IsPeriodicSpaceGroup(SpaceGroup(4,k)));
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23, 25,
  26, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 42, 43, 44, 46, 47, 48,
  49, 50, 51, 52, 53, 54, 55, 56, 58, 59, 60, 61, 62, 63, 65, 66, 68, 69, 70,
  71, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 89, 90, 91,
  93, 94, 95, 96, 97, 98, 99, 101, 102, 103, 104, 105, 107, 108, 109, 110,
  111, 113, 115, 116, 118, 119, 120, 121, 122, 124, 126, 127, 128, 130, 131,
  134, 141, 144, 145, 149, 151, 153, 154, 155, 156, 157, 158, 159, 160, 162,
  163, 165, 167, 168, 169, 170, 171, 172, 173, 174, 176, 178, 179, 180, 187,
  188, 197, 198, 202, 204, 205, 206, 211, 212, 219, 220, 222, 226, 233, 237,
  238, 239, 240, 241, 242, 243, 244, 245, 247, 248, 249, 250, 251, 253, 254,
  255, 256, 257, 259, 260, 261, 263, 264, 265, 266, 267, 269, 270, 271, 273,
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275, 277, 278, 279, 281, 283, 285, 290, 292, 296, 297, 298, 299, 300, 301,
303, 304, 305, 314, 316, 317, 319, 327, 328, 329, 333, 335, 342, 355, 357,
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505, 507, 508, 512, 514, 515, 516, 517, 522, 524, 525, 526, 527, 533, 537,
539, 540, 541, 542, 543, 544, 546, 548, 553, 555, 558, 562, 564, 565, 566,
567, 568, 571, 572, 573, 574, 576, 577, 580, 581, 582, 589, 590, 591, 593,
596, 598, 599, 612, 613, 622, 623, 624, 626, 632, 641, 647, 649, 651, 652,
654, 656, 657, 658, 659, 661, 662, 663, 665, 666, 667, 668, 669, 670, 671,
672, 674, 676, 677, 678, 679, 680, 682, 683, 684, 686, 688, 689, 690, 691,
692, 694, 696, 697, 698, 699, 700, 702, 708, 710, 712, 714, 716, 720, 722,
728, 734, 738, 739, 741, 742, 744, 745, 752, 754, 756, 757, 758, 762, 763,
769, 770, 778, 779, 784, 788, 790, 800, 801, 843, 845, 854, 855, 856, 857,
865, 874, 900, 904, 909, 911, 913, 915, 916, 917, 919, 920, 921, 922, 923,
924, 925, 926, 927, 929, 931, 932, 933, 934, 936, 938, 940, 941, 943, 945,
946, 953, 955, 956, 958, 963, 966, 972, 973, 978, 979, 981, 982, 983, 985,
987, 988, 989, 991, 992, 993, 995, 996, 998, 999, 1000, 1003, 1011, 1022,
1024, 1025, 1026, 1162, 1167, 1236, 1237, 1238, 1239, 1240, 1241, 1242,
1243, 1244, 1246, 1248, 1250, 1255, 1264, 1267, 1270, 1273, 1279, 1280,
1281, 1283, 1284, 1289, 1291, 1293, 1294, 1324, 1325, 1326, 1327, 1328,
1329, 1330, 1331, 1332, 1333, 1334, 1335, 1336, 1337, 1338, 1339, 1340,
1341, 1343, 1345, 1347, 1348, 1349, 1350, 1351, 1352, 1354, 1356, 1357,
1358, 1359, 1361, 1363, 1365, 1367, 1372, 1373, 1374, 1375, 1376, 1377,
1378, 1379, 1380, 1381, 1382, 1383, 1384, 1385, 1386, 1387, 1388, 1389,
1390, 1393, 1395, 1397, 1399, 1400, 1401, 1404, 1405, 1408, 1410, 1419,
1420, 1421, 1422, 1424, 1425, 1426, 1428, 1429, 1438, 1440, 1441, 1442,
1443, 1444, 1445, 1449, 1450, 1451, 1456, 1457, 1460, 1461, 1462, 1464,
1465, 1470, 1472, 1473, 1477, 1480, 1481, 1487, 1488, 1489, 1493, 1494,
1495, 1501, 1503, 1506, 1509, 1512, 1515, 1518, 1521, 1524, 1527, 1530,
1532, 1533, 1534, 1537, 1538, 1541, 1542, 1544, 1547, 1550, 1552, 1553,
1554, 1558, 1565, 1566, 1568, 1573, 1644, 1648, 1673, 1674, 1700, 1702,
1705, 1713, 1714, 1735, 1738, 1740, 1741, 1742, 1743, 1744, 1745, 1746,
1747, 1748, 1749, 1750, 1751, 1752, 1753, 1754, 1755, 1756, 1757, 1759,
1761, 1762, 1763, 1765, 1767, 1768, 1769, 1770, 1771, 1772, 1773, 1774,
1775, 1778, 1779, 1782, 1783, 1785, 1787, 1788, 1789, 1791, 1793, 1795,
1797, 1798, 1799, 1800, 1801, 1803, 1806, 1807, 1809, 1810, 1811, 1813,
1815, 1821, 1822, 1823, 1828, 1829, 1833, 1837, 1839, 1842, 1845, 1848,
1850, 1851, 1852, 1854, 1856, 1857, 1858, 1859, 1860, 1861, 1863, 1866,
1870, 1873, 1874, 1877, 1880, 1883, 1885, 1886, 1887, 1889, 1892, 1895,
1915, 1918, 1920, 1923, 1925, 1927, 1928, 1930, 1952, 1953, 1954, 1955,
2045, 2047, 2049, 2051, 2053, 2054, 2055, 2056, 2057, 2059, 2067, 2068,
2072, 2075, 2076, 2079, 2084, 2087, 2088, 2092, 2133, 2135, 2136, 2137,
2139, 2140, 2170, 2171, 2196, 2224, 2234, 2236, 2238, 2254, 2355, 2356,
2386, 2387, 2442, 2445, 2448, 2451, 2478, 2484, 2487, 2490, 2493, 2496,
2499, 2502, 2508, 2511, 2514, 2517, 2520, 2523, 2550, 2553, 2559, 2621,
2624, 2648, 2650, 3046, 3047, 3048, 3049, 3050, 3051, 3052, 3053, 3054,
3055, 3056, 3057, 3058, 3059, 3060, 3061, 3062, 3063, 3064, 3065, 3066,
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    3067, 3068, 3069, 3070, 3071, 3072, 3073, 3074, 3075, 3076, 3077, 3078,
    3079, 3080, 3081, 3082, 3083, 3084, 3085, 3086, 3087, 3089, 3090, 3091,
    3094, 3095, 3096, 3099, 3100, 3101, 3104, 3105, 3106, 3109, 3110, 3111,
    3112, 3113, 3114, 3115, 3117, 3119, 3120, 3121, 3122, 3123, 3124, 3125,
    3127, 3128, 3129, 3130, 3131, 3132, 3133, 3135, 3137, 3139, 3141, 3142,
    3143, 3144, 3145, 3149, 3151, 3152, 3153, 3154, 3155, 3157, 3159, 3160,
    3161, 3162, 3163, 3169, 3170, 3171, 3172, 3173, 3174, 3175, 3177, 3179,
    3180, 3181, 3182, 3183, 3184, 3185, 3187, 3188, 3189, 3190, 3191, 3192,
    3193, 3195, 3197, 3199, 3200, 3201, 3204, 3206, 3207, 3208, 3209, 3210,
    3212, 3214, 3215, 3216, 3217, 3218, 3226, 3234, 3235, 3236, 3244, 3252,
    3253, 3254, 3260, 3268, 3269, 3270, 3278, 3280, 3281, 3282, 3283, 3284,
    3285, 3286, 3287, 3288, 3289, 3290, 3291, 3292, 3295, 3296, 3298, 3299,
    3302, 3303, 3306, 3308, 3309, 3310, 3311, 3312, 3313, 3314, 3315, 3316,
    3317, 3318, 3319, 3320, 3322, 3324, 3326, 3327, 3329, 3330, 3338, 3345,
    3346, 3347, 3348, 3350, 3351, 3352, 3354, 3355, 3356, 3359, 3360, 3361,
    3362, 3374, 3375, 3383, 3385, 3398, 3399, 3417, 3418, 3419, 3420, 3422,
    3424, 3426, 3428, 3446, 3447, 3455, 3457, 3469, 3471, 3521, 3523, 3524,
    3525, 3530, 3531, 3534, 3539, 3542, 3545, 3548, 3550, 3551, 3554, 3557,
    3579, 3580, 3830, 3831, 3832, 3833, 3835, 3837, 3839, 3849, 3851, 3877,
    3938, 3939, 3949, 3951, 3952, 3958, 3960, 3962, 3963, 3964, 3966, 3968,
    3972, 3973, 3975, 4006, 4029, 4030, 4033, 4034, 4037, 4038, 4046, 4048,
    4050, 4062, 4064, 4067, 4078, 4081, 4089, 4090, 4114, 4138, 4139, 4140,
    4141, 4146, 4147, 4148, 4149, 4154, 4155, 4169, 4171, 4175, 4180, 4183,
    4188, 4190, 4204, 4205, 4223, 4224, 4225, 4254, 4286, 4289, 4391, 4397,
    4496, 4499, 4500, 4501, 4502, 4504, 4508, 4510, 4521, 4525, 4544, 4559,
    4560, 4561, 4562, 4579, 4580, 4581, 4583, 4587, 4597, 4598, 4599, 4600,
    4651, 4759, 4760, 4761, 4762, 4766 ]
```

Chapter 2

Cubical complexes & permutahedral complexes

2.1 Cubical complexes

A *finite simplicial complex* can be defined to be a CW-subcomplex of the canonical regular CW-structure on a simplex Δ^n of some dimension n. Analogously, a *finite cubical complex* is a CW-subcomplex of the regular CW-structure on a cube $[0,1]^n$ of some dimension n. Equivalently, but more conveniently, we can replace the unit interval $[0,1]$ by an interval $[0,k]$ with CW-structure involving $2k+1$ cells, namely one 0-cell for each integer $0 \leq j \leq k$ and one 1-cell for each open interval $(j, j+1)$ for $0 \leq j \leq k-1$. A *finite cuical complex M* is a CW-subcompex $M \subset [0,k_1] \times [0,k_2] \times \cdots [0,k_n]$ of a direct product of intervals, the direct product having the usual direct product CW-structure. The equivalence of these two definitions follows from the Gray code embedding of a mesh into a hypercube. We say that the cubical complex has *ambient dimension n*. A cubical complex M of ambient dimension n is said to be *pure* if each cell lies in the boundary of an n-cell. In other words, M is pure if it is a union of unit n-cubes in $\mathbb{R}^n$, each unit cube having vertices with integer coordinates.

HAP has a datatype for finite cubical complexes, and a slightly different datatype for pure cubical complexes.

The following example constructs the granny knot (the sum of a trefoil knot with its reflection) as a 3-dimensional pure cubical complex, and then displays it.

```
──────────────────────── Example ────────────────────────
gap> K:=PureCubicalKnot(3,1);
prime knot 1 with 3 crossings

gap> L:=ReflectedCubicalKnot(K);
Reflected( prime knot 1 with 3 crossings )

gap> M:=KnotSum(K,L);
prime knot 1 with 3 crossings + Reflected( prime knot 1 with 3 crossings )

gap> Display(M);
```

Next we construct the complement $Y = D^3 \setminus \mathring{M}$ of the interior of the pure cubical complex M. Here D^3 is a rectangular region with $M \subset \mathring{D}^3$. This pure cubical complex Y is a union of 5891 unit

3-cubes. We contract *Y* to get a homotopy equivalent pure cubical complex *YY* consisting of the union of just 775 unit 3-cubes. Then we convert *YY* to a regular CW-complex *W* involving 11939 cells. We contract *W* to obtain a homotopy equivalent regular CW-complex *WW* involving 5993 cells. Finally we compute the fundamental group of the complement of the granny knot, and use the presentation of this group to establish that the Alexander polynomial *P*(*x*) of the granny is

$P(x) = x^4 - 2x^3 + 3x^2 - 2x + 1$.

─────── Example ───────

```
gap> Y:=PureComplexComplement(M);
Pure cubical complex of dimension 3.

gap> Size(Y);
5891

gap> YY:=ZigZagContractedComplex(Y);
Pure cubical complex of dimension 3.

gap> Size(YY);
775

gap> W:=RegularCWComplex(YY);
Regular CW-complex of dimension 3

gap> Size(W);
11939

gap> WW:=ContractedComplex(W);
Regular CW-complex of dimension 2

gap> Size(WW);
5993

gap> G:=FundamentalGroup(WW);
<fp group of size infinity on the generators [ f1, f2, f3 ]>

gap> AlexanderPolynomial(G);
x_1^4-2*x_1^3+3*x_1^2-2*x_1+1
```

2.2 Permutahedral complexes

A finite pure cubical complex is a union of finitely many cubes in a tessellation of $\mathbb{R}^n$ by unit cubes. One can also tessellate $\mathbb{R}^n$ by permutahedra, and we define a finite *n*-dimensional pure *permutahedral complex* to be a union of finitely many permutahdra from such a tessellation. There are two features of pure permutahedral complexes that are particularly useful in some situations:

- Pure permutahedral complexes are topological manifolds with boundary.

- The method used for finding a smaller pure cubical complex *M'* homotopy equivalent to a given pure cubical complex *M* retains the homeomorphism type, and not just the homotopy type, of the space *M*.

EXAMPLE 1

To illustrate these features the following example begins by reading in a protein backbone from the online Protein Database, and storing it as a pure cubical complex K. The ends of the protein have been joined, and the homology $H_i(K, \mathbb{Z}) = \mathbb{Z}$, $i = 0, 1$ is seen to be that of a circle. We can thus regard the protein as a knot $K \subset \mathbb{R}^3$. The protein is visualized as a pure permutahedral complex.

```
————————————————— Example —————————————————
  gap> file:=HapFile("data1V2X.pdb");;
  gap> K:=ReadPDBfileAsPurePermutahedralComplex("file");
  Pure permutahedral complex of dimension 3.

  gap> Homology(K,0);
  [ 0 ]
  gap> Homology(K,1);
  [ 0 ]

  Display(K);
```

An alternative method for seeing that the pure permutahedral complex K has the homotopy type of a circle is to note that it is covered by open permutahedra (small open neighbourhoods of the closed 3-dimensional permutahedral titles) and to form the nerve $N = Nerve(\mathscr{U})$ of this open covering $\mathscr{U}$. The nerve N has the same homotopy type as K. The following commands establish that N is a 1-dimensional simplicial complex and display N as a circular graph.

```
————————————————— Example —————————————————
  gap> N:=Nerve(K);
  Simplicial complex of dimension 1.

  gap> Display(GraphOfSimplicialComplex(N));
```

The boundary of the pure permutahedral complex K is a 2-dimensional CW-complex B homeomorphic to a torus. We next use the advantageous features of pure permutahedral complexes to compute the homomorphism

$$\phi \colon \pi_1(B) \to \pi_1(\mathbb{R}^3 \setminus \mathring{K}), a \mapsto yx^{-3}y^2x^{-2}yxy^{-1}, b \mapsto yx^{-1}y^{-1}x^2y^{-1}$$

where

$\pi_1(B) = <a, b : aba^{-1}b^{-1} = 1>$,
$\pi_1(\mathbb{R}^3 \setminus \mathring{K}) \cong <x, y : y^2x^{-2}yxy^{-1} = 1, yx^{-2}y^{-1}x(xy^{-1})^2 = 1>$.

```
————————————————— Example —————————————————
  gap> Y:=PureComplexComplement(K);
  Pure permutahedral complex of dimension 3.
  gap> Size(Y);
  418922

  gap> YY:=ZigZagContractedComplex(Y);
  Pure permutahedral complex of dimension 3.
  gap> Size(YY);
  3438

  gap> W:=RegularCWComplex(YY);
  Regular CW-complex of dimension 3
```

```
gap> f:=BoundaryMap(W);
Map of regular CW-complexes

gap> CriticalCells(Source(f));
[ [ 2, 1 ], [ 2, 261 ], [ 1, 1043 ], [ 1, 1626 ], [ 0, 2892 ], [ 0, 24715 ] ]

gap> F:=FundamentalGroup(f,2892);
[ f1, f2 ] -> [ f2*f1^-3*f2^2*f1^-2*f2*f1*f2^-1, f2*f1^-1*f2^-1*f1^2*f2^-1 ]

gap> G:=Target(F);
<fp group on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(G);
[ f2^2*f1^-2*f2*f1*f2^-1, f2*f1^-2*f2^-1*f1*(f1*f2^-1)^2 ]
```

EXAMPLE 2

The next example of commands begins by readng two synthetic knots from a CSV file (containing the coordinates of the two sequences of vertices) and producing a pure permutahedral complex model of the two knots. The linking number of two knots is given by the low-dimension cup product of the complement of the knots. This linking number is computed to be 6.

```
───────────────────── Example ─────────────────────
gap> file1:=HapFile("data175_1.csv");;
gap> file2:=HapFile("data175_2.csv");;
gap> K:=ReadCSVfileAsPureCubicalKnot( [file1, file2]);;
gap> K:=PurePermutahedralComplex(K!.binaryArray);;
gap> K:=ThickenedPureComplex(K);;
gap> K:=ContractedComplex(K);;
gap> #K is a permutahedral complex model of the two input knots
gap> Display(K);


gap> Y:=PureComplexComplement(K);;
gap> W:=ZigZagContractedComplex(Y,2);;
gap> W:=RegularCWComplex(W);;
gap> W:=ContractedComplex(W);;
gap> G:=FundamentalGroup(W);;
gap> cup:=CupProduct(G);;
gap> cup([1,0],[0,1]);
[ -6, 0 ]
```

2.3 Constructing pure cubical and permutahedral complexes

An *n*-dimensional pure cubical or permutahedral complex can be created from an *n*-dimensional array of 0s and 1s. The following example creates and displays two 3-dimensional complexes.

```
───────────────────── Example ─────────────────────
gap> A:=[[[0,0,0],[0,0,0],[0,0,0]],
>        [[1,1,1],[1,0,1],[1,1,1]],
```

```
>            [[0,0,0],[0,0,0],[0,0,0]]];;
gap> M:=PureCubicalComplex(A);
Pure cubical complex of dimension 3.

gap> P:=PurePermutahedralComplex(A);
Pure permutahedral complex of dimension 3.

gap> Display(M);
gap> Display(P);
```

2.4 Computations in dynamical systems

Pure cubical complexes can be useful for rigourous interval arithmetic calculations in numerical analysis. They can also be useful for trying to estimate approximations of certain numerical quantities. To illustrate the latter we consider the *Henon map*

$$f:\mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y+1-ax^2 \\ bx \end{pmatrix}.$$

Starting with $(x_0, y_0) = (0,0)$ and iterating $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$ with the parameter values $a = 1.4$, $b = 0.3$ one obtains a sequence of points which is known to be dense in the so called *strange attractor* $\mathscr{A}$ of the Henon map. The first 10 million points in this sequence are plotted in the following example, with arithmetic performed to 100 decimal places of accuracy. The sequence is stored as a 2-dimensional pure cubical complex where each 2-cell is square of side equal to $\varepsilon = 1/500$.

```
───────────────────────── Example ─────────────────────────
gap> M:=HenonOrbit([0,0],14/10,3/10,10^7,500,100);
Pure cubical complex of dimension 2.

gap> Size(M);
10287

gap> Display(M);
```

Repeating the computation but with squares of side $\varepsilon = 1/1000$

```
───────────────────────── Example ─────────────────────────
gap> M:=HenonOrbit([0,0],14/10,3/10,10^7,1000,100);

gap> Size(M);
24949
```

we obtain the heuristic estimate

$$\delta \simeq \frac{\log 24949 - \log 10287}{\log 2} = 1.277$$

for the box-counting dimension of the attractor $\mathscr{A}$.

Chapter 3

Covering spaces

Let Y denote a finite regular CW-complex. Let $\widetilde{Y}$ denote its universal covering space. The covering space inherits a regular CW-structure which can be computed and stored using the datatype of a $\pi_1 Y$-equivariant CW-complex. The cellular chain complex $C_*\widetilde{Y}$ of $\widetilde{Y}$ can be computed and stored as an equivariant chain complex. Given an admissible discrete vector field on Y, we can endow Y with a smaller non-regular CW-structre whose cells correspond to the critical cells in the vector field. This smaller CW-structure leads to a more efficient chain complex $C_*\widetilde{Y}$ involving one free generator for each critical cell in the vector field.

3.1 Cellular chains on the universal cover

The following commands construct a 6-dimensional regular CW-complex $Y \simeq S^1 \times S^1 \times S^1$ homotopy equivalent to a product of three circles.

```
———————————————————————— Example ————————————————————————
  gap> A:=[[1,1,1],[1,0,1],[1,1,1]];;
  gap> S:=PureCubicalComplex(A);;
  gap> T:=DirectProduct(S,S,S);;
  gap> Y:=RegularCWComplex(T);;
  Regular CW-complex of dimension 6

  gap> Size(Y);
  110592
```

The CW-somplex Y has 110592 cells. The next commands construct a free $\pi_1 Y$-equivariant chain complex $C_*\widetilde{Y}$ homotopy equivalent to the chain complex of the universal cover of Y. The chain complex $C_*\widetilde{Y}$ has just 8 free generators.

```
———————————————————————— Example ————————————————————————
  gap> Y:=ContractedComplex(Y);;
  gap> CU:=ChainComplexOfUniversalCover(Y);;
  gap> List([0..Dimension(Y)],n->CU!.dimension(n));
  [ 1, 3, 3, 1 ]
```

The next commands construct a subgroup $H < \pi_1 Y$ of index 50 and the chain complex $C_*\widetilde{Y} \otimes_{\mathbb{Z}H} \mathbb{Z}$ which is homotopy equivalent to the cellular chain complex $C_*\widetilde{Y}_H$ of the 50-fold cover $\widetilde{Y}_H$ of Y corresponding to H.

```
―――――――――――――――――― Example ――――――――――――――――――
gap> L:=LowIndexSubgroupsFpGroup(CU!.group,50);;
gap> H:=L[Length(L)-1];;
gap> Index(CU!.group,H);
50
gap> D:=TensorWithIntegersOverSubgroup(CU,H);
Chain complex of length 3 in characteristic 0 .

gap> List([0..3],D!.dimension);
[ 50, 150, 150, 50 ]
```

General theory implies that the 50-fold covering space $\widetilde{Y}_H$ should again be homotopy equivalent to a product of three circles. In keeping with this, the following commands verify that $\widetilde{Y}_H$ has the same integral homology as $S^1 \times S^1 \times S^1$.

```
―――――――――――――――――― Example ――――――――――――――――――
gap> Homology(D,0);
[ 0 ]
gap> Homology(D,1);
[ 0, 0, 0 ]
gap> Homology(D,2);
[ 0, 0, 0 ]
gap> Homology(D,3);
[ 0 ]
```

3.2 Spun knots and the Satoh tube map

We'll contruct two spaces Y, W with isomorphic fundamental groups and isomorphic intergal homology, and use the integral homology of finite covering spaces to establsh that the two spaces have distinct homotopy types.

By *spinning* a link $K \subset \mathbb{R}^3$ about a plane $P \subset \mathbb{R}^3$ with $P \cap K = \emptyset$, we obtain a collection $Sp(K) \subset \mathbb{R}^4$ of knotted tori. The following commands produce the two tori obtained by spinning the Hopf link K and show that the space $Y = \mathbb{R}^4 \setminus Sp(K) = Sp(\mathbb{R}^3 \setminus K)$ is connected with fundamental group $\pi_1 Y = \mathbb{Z} \times \mathbb{Z}$ and homology groups $H_0(Y) = \mathbb{Z}$, $H_1(Y) = \mathbb{Z}^2$, $H_2(Y) = \mathbb{Z}^4$, $H_3(Y, \mathbb{Z}) = \mathbb{Z}^2$. The space Y is only constructed up to homotopy, and for this reason is 3-dimensional.

```
―――――――――――――――――― Example ――――――――――――――――――
gap> Hopf:=PureCubicalLink("Hopf");
Pure cubical link.

gap> Y:=SpunAboutInitialHyperplane(PureComplexComplement(Hopf));
Regular CW-complex of dimension 3

gap> Homology(Y,0);
[ 0 ]
gap> Homology(Y,1);
[ 0, 0 ]
gap> Homology(Y,2);
[ 0, 0, 0, 0 ]
gap> Homology(Y,3);
```

```
[ 0, 0 ]
gap> Homology(Y,4);
[ ]
gap> GY:=FundamentalGroup(Y);;
gap> GeneratorsOfGroup(GY);
[ f2, f3 ]
gap> RelatorsOfFpGroup(GY);
[ f3^-1*f2^-1*f3*f2 ]
```

An alternative embedding of two tori $L \subset \mathbb{R}^4$ can be obtained by applying the 'tube map' of Shin Satoh to a welded Hopf link [Sat00]. The following commands construct the complement $W = \mathbb{R}^4 \setminus L$ of this alternative embedding and show that W has the same fundamental group and integral homology as Y above.

```
                              ──── Example ────
gap> L:=HopfSatohSurface();
Pure cubical complex of dimension 4.

gap> W:=ContractedComplex(RegularCWComplex(PureComplexComplement(L)));
Regular CW-complex of dimension 3

gap> Homology(W,0);
[ 0 ]
gap> Homology(W,1);
[ 0, 0 ]
gap> Homology(W,2);
[ 0, 0, 0, 0 ]
gap> Homology(W,3);
[ 0, 0 ]
gap> Homology(W,4);
[ ]

gap> GW:=FundamentalGroup(W);;
gap> GeneratorsOfGroup(GW);
[ f1, f2 ]
gap> RelatorsOfFpGroup(GW);
[ f1^-1*f2^-1*f1*f2 ]
```

Despite having the same fundamental group and integral homology groups, the above two spaces Y and W were shown by Kauffman and Martins [KFM08] to be not homotopy equivalent. Their technique involves the fundamental crossed module derived from the first three dimensions of the universal cover of a space, and counts the representations of this fundamental crossed module into a given finite crossed module. This homotopy inequivalence is recovered by the following commands which involves the 5-fold covers of the spaces.

```
                              ──── Example ────
gap> CY:=ChainComplexOfUniversalCover(Y);
Equivariant chain complex of dimension 3
gap> LY:=LowIndexSubgroups(CY!.group,5);;
gap> invY:=List(LY,g->Homology(TensorWithIntegersOverSubgroup(CY,g),2));;
```

```
gap> CW:=ChainComplexOfUniversalCover(W);
Equivariant chain complex of dimension 3
gap> LW:=LowIndexSubgroups(CW!.group,5);;
gap> invW:=List(LW,g->Homology(TensorWithIntegersOverSubgroup(CW,g),2));;

gap> SSortedList(invY)=SSortedList(invW);
false
```

3.3 Cohomology with local coefficients

The $\pi_1 Y$-equivariant cellular chain complex $C_* \widetilde{Y}$ of the universal cover $\widetilde{Y}$ of a regular CW-complex Y can be used to compute the homology $H_n(Y, A)$ and cohomology $H^n(Y, A)$ of Y with local coefficients in a $\mathbb{Z}\pi_1 Y$-module A. To illustrate this we consister the space Y arising as the complement of the trefoil knot, with fundamental group $\pi_1 Y = \langle x, y : xyx = yxy \rangle$. We take $A = \mathbb{Z}$ to be the integers with non-trivial $\pi_1 Y$-action given by $x.1 = -1, y.1 = -1$. We then compute

$$
\begin{aligned}
H_0(Y, A) &= \mathbb{Z}_2, \\
H_1(Y, A) &= \mathbb{Z}_3, \\
H_2(Y, A) &= \mathbb{Z}.
\end{aligned}
$$

```
——— Example ———
gap> K:=PureCubicalKnot(3,1);;
gap> Y:=PureComplexComplement(K);;
gap> Y:=ContractedComplex(Y);;
gap> Y:=RegularCWComplex(Y);;
gap> Y:=SimplifiedComplex(Y);;
gap> C:=ChainComplexOfUniversalCover(Y);;
gap> G:=C!.group;;
gap> GeneratorsOfGroup(G);
[ f1, f2 ]
gap> RelatorsOfFpGroup(G);
[ f2^-1*f1^-1*f2^-1*f1*f2*f1, f1^-1*f2^-1*f1^-1*f2*f1*f2 ]
gap> hom:=GroupHomomorphismByImages(G,Group([[-1]]),[G.1,G.2],[[[-1]],[[-1]]]);;
gap> A:=function(x); return Determinant(Image(hom,x)); end;;
gap> D:=TensorWithTwistedIntegers(C,A); #Here the function A represents
gap> #the integers with twisted action of G.
Chain complex of length 3 in characteristic 0 .
gap> Homology(D,0);
[ 2 ]
gap> Homology(D,1);
[ 3 ]
gap> Homology(D,2);
[ 0 ]
```

3.4   Distinguishing between two non-homeomorphic homotopy equivalent spaces

The granny knot is the sum of the trefoil knot and its mirror image. The reef knot is the sum of two identical copies of the trefoil knot. The following commands show that the degree 1 homology homomorphisms

$H_1(p^{-1}(B), \mathbb{Z}) \to H_1(\widetilde{X}_H, \mathbb{Z})$

distinguish between the homeomorphism types of the complements $X \subset \mathbb{R}^3$ of the granny knot and the reef knot, where $B \subset X$ is the knot boundary, and where $p : \widetilde{X}_H \to X$ is the covering map corresponding to the finite index subgroup $H < \pi_1 X$. More precisely, $p^{-1}(B)$ is in general a union of path components

$p^{-1}(B) = B_1 \cup B_2 \cup \cdots \cup B_t$.

The function `FirstHomologyCoveringCokernels(f,c)` inputs an integer c and the inclusion $f : B \hookrightarrow X$ of a knot boundary B into the knot complement X. The function returns the ordered list of the lists of abelian invariants of cokernels

coker($H_1(p^{-1}(B_i), \mathbb{Z}) \to H_1(\widetilde{X}_H, \mathbb{Z})$)

arising from subgroups $H < \pi_1 X$ of index c. To distinguish between the granny and reef knots we use index $c = 6$.

```
──────────────────────── Example ────────────────────────
 gap> K:=PureCubicalKnot(3,1);;
 gap> L:=ReflectedCubicalKnot(K);;
 gap> granny:=KnotSum(K,L);;
 gap> reef:=KnotSum(K,K);;
 gap> fg:=KnotComplementWithBoundary(ArcPresentation(granny));;
 gap> fr:=KnotComplementWithBoundary(ArcPresentation(reef));;
 gap> a:=FirstHomologyCoveringCokernels(fg,6);;
 gap> b:=FirstHomologyCoveringCokernels(fr,6);;
 gap> a=b;
 false
```

3.5   Second homotopy groups of spaces with finite fundamental group

If $p : \widetilde{Y} \to Y$ is the universal covering map, then the fundamental group of $\widetilde{Y}$ is trivial and the Hurewicz homomorphism $\pi_2 \widetilde{Y} \to H_2(\widetilde{Y}, \mathbb{Z})$ from the second homotopy group of $\widetilde{Y}$ to the second integral homology of $\widetilde{Y}$ is an isomorphism. Furthermore, the map p induces an isomorphism $\pi_2 \widetilde{Y} \to \pi_2 Y$. Thus $H_2(\widetilde{Y}, \mathbb{Z})$ is isomorphic to the second homotopy group $\pi_2 Y$.

If the fundamental group of Y happens to be finite, then in principle we can calculate $H_2(\widetilde{Y}, \mathbb{Z}) \cong \pi_2 Y$. We illustrate this computation for Y equal to the real projective plane. The above computation shows that Y has second homotopy group $\pi_2 Y \cong \mathbb{Z}$.

```
──────────────────────── Example ────────────────────────
 gap> K:=[ [1,2,3], [1,3,4], [1,2,6], [1,5,6], [1,4,5],
 >         [2,3,5], [2,4,5], [2,4,6], [3,4,6], [3,5,6]];;

 gap> K:=MaximalSimplicesToSimplicialComplex(K);
 Simplicial complex of dimension 2.

 gap> Y:=RegularCWComplex(K);
```

```
Regular CW-complex of dimension 2
gap> # Y is a regular CW-complex corresponding to the projective plane.

gap> U:=UniversalCover(Y);
Equivariant CW-complex of dimension 2

gap> G:=U!.group;;
gap> # G is the fundamental group of Y, which by the next command
gap> # is finite of order 2.
gap> Order(G);
2

gap> U:=EquivariantCWComplexToRegularCWComplex(U,Group(One(G)));
Regular CW-complex of dimension 2
gap> #U is the universal cover of Y

gap> Homology(U,0);
[ 0 ]
gap> Homology(U,1);
[  ]
gap> Homology(U,2);
[ 0 ]
```

3.6   Third homotopy groups of simply connected spaces

3.6.1   First example: Whitehead's certain exact sequence

For any path connected space Y with universal cover $\widetilde{Y}$ there is an exact sequence
$$\to \pi_4\widetilde{Y} \to H_4(\widetilde{Y},\mathbb{Z}) \to H_4(K(\pi_2\widetilde{Y},2),\mathbb{Z}) \to \pi_3\widetilde{Y} \to H_3(\widetilde{Y},\mathbb{Z}) \to 0$$
   due to J.H.C.Whitehead. Here $K(\pi_2(\widetilde{Y}),2)$ is an Eilenberg-MacLane space with second homotopy group equal to $\pi_2\widetilde{Y}$.

   Continuing with the above example where Y is the real projective plane, we see that $H_4(\widetilde{Y},\mathbb{Z}) = H_3(\widetilde{Y},\mathbb{Z}) = 0$ since $\widetilde{Y}$ is a 2-dimensional CW-space. The exact sequence implies $\pi_3\widetilde{Y} \cong H_4(K(\pi_2\widetilde{Y},2),\mathbb{Z})$. Furthermore, $\pi_3\widetilde{Y} = \pi_3Y$. The following commands establish that $\pi_3Y \cong \mathbb{Z}$.

──────── Example ────────
```
gap> A:=AbelianPcpGroup([0]);
Pcp-group with orders [ 0 ]

gap> K:=EilenbergMacLaneSimplicialGroup(A,2,5);;
gap> C:=ChainComplexOfSimplicialGroup(K);
Chain complex of length 5 in characteristic 0 .

gap> Homology(C,4);
[ 0 ]
```

3.6.2 Second example: the Hopf invariant

The following commands construct a 4-dimensional simplicial complex Y with 9 vertices and 36 4-dimensional simplices, and establish that
$$\pi_1 Y = 0, \pi_2 Y = \mathbb{Z}, H_3(Y, \mathbb{Z}) = 0, H_4(Y, \mathbb{Z}) = \mathbb{Z}.$$

```
                          Example
gap> smp:=[ [ 1, 2, 4, 5, 6 ], [ 1, 2, 4, 5, 9 ], [ 1, 2, 5, 6, 8 ],
>        [ 1, 2, 6, 4, 7 ], [ 2, 3, 4, 5, 8 ], [ 2, 3, 5, 6, 4 ],
>        [ 2, 3, 5, 6, 7 ], [ 2, 3, 6, 4, 9 ], [ 3, 1, 4, 5, 7 ],
>        [ 3, 1, 5, 6, 9 ], [ 3, 1, 6, 4, 5 ], [ 3, 1, 6, 4, 8 ],
>        [ 4, 5, 7, 8, 3 ], [ 4, 5, 7, 8, 9 ], [ 4, 5, 8, 9, 2 ],
>        [ 4, 5, 9, 7, 1 ], [ 5, 6, 7, 8, 2 ], [ 5, 6, 8, 9, 1 ],
>        [ 5, 6, 8, 9, 7 ], [ 5, 6, 9, 7, 3 ], [ 6, 4, 7, 8, 1 ],
>        [ 6, 4, 8, 9, 3 ], [ 6, 4, 9, 7, 2 ], [ 6, 4, 9, 7, 8 ],
>        [ 7, 8, 1, 2, 3 ], [ 7, 8, 1, 2, 6 ], [ 7, 8, 2, 3, 5 ],
>        [ 7, 8, 3, 1, 4 ], [ 8, 9, 1, 2, 5 ], [ 8, 9, 2, 3, 1 ],
>        [ 8, 9, 2, 3, 4 ], [ 8, 9, 3, 1, 6 ], [ 9, 7, 1, 2, 4 ],
>        [ 9, 7, 2, 3, 6 ], [ 9, 7, 3, 1, 2 ], [ 9, 7, 3, 1, 5 ] ];;

gap> K:=MaximalSimplicesToSimplicialComplex(smp);
Simplicial complex of dimension 4.

gap> Y:=RegularCWComplex(Y);
Regular CW-complex of dimension 4

gap> Order(FundamentalGroup(Y));
1
gap> Homology(Y,2);
[ 0 ]
gap> Homology(Y,3);
[ ]
gap> Homology(Y,4);
[ 0 ]
```

Previous commands have established $H_4(K(\mathbb{Z}, 2), \mathbb{Z}) = \mathbb{Z}$. So Whitehead's sequence reduces to an exact sequence
$$\mathbb{Z} \to \mathbb{Z} \to \pi_3 Y \to 0$$
in which the first map is $H_4(Y, \mathbb{Z}) = \mathbb{Z} \to H_4(K(\pi_2 Y, 2), \mathbb{Z}) = \mathbb{Z}$. Hence $\pi_3 Y$ is cyclic.

HAP is currently unable to compute the order of $\pi_3 Y$ directly from Whitehead's sequence. Instead, we can use the *Hopf invariant*. For any map $\phi \colon S^3 \to S^2$ we consider the space $C(\phi) = S^2 \cup_\phi e^4$ obtained by attaching a 4-cell e^4 to S^2 via the attaching map ϕ. The cohomology groups $H^2(C(\phi), \mathbb{Z}) = \mathbb{Z}$, $H^4(C(\phi), \mathbb{Z}) = \mathbb{Z}$ are generated by elements α, β say, and the cup product has the form
$$- \cup - \colon H^2(C(\phi), \mathbb{Z}) \times H^2(C(\phi), \mathbb{Z}) \to H^4(C(\phi), \mathbb{Z}), (\alpha, \alpha) \mapsto h_\phi \beta$$
for some integer h_ϕ. The integer h_ϕ is the HOPF INVARIANT. The function $h \colon \pi_3(S^3) \to \mathbb{Z}$ is a homomorphism and there is an isomorphism
$$\pi_3(S^2 \cup e^4) \cong \mathbb{Z}/\langle h_\phi \rangle.$$

The following commands begin by simplifying the cell structure on the above CW-complex $Y \cong K$ to obtain a homeomorphic CW-complex W with fewer cells. They then create a space S by removing one 4-cell from W. The space S is seen to be homotopy equivalent to a CW-complex $e^2 \cup e^0$ with a

single 0-cell and single 2-cell. Hence $S \simeq S^2$ is homotopy equivalent to the 2-sphere. Consequently $Y \simeq C(\phi) = S^2 \cup_\phi e^4$ for some map $\phi: S^3 \to S^2$.

```
———————————————————— Example ————————————————————
  gap> W:=SimplifiedComplex(Y);
  Regular CW-complex of dimension 4

  gap> S:=RegularCWComplexWithRemovedCell(W,4,6);
  Regular CW-complex of dimension 4

  gap> CriticalCells(S);
  [ [ 2, 6 ], [ 0, 5 ] ]
```

The next commands show that the map ϕ in the construction $Y \simeq C(\phi)$ has Hopf invariant -1. Hence $h: \pi_3(S^3) \to \mathbb{Z}$ is an isomorphism. Therefore $\pi_3 Y = 0$.

```
———————————————————— Example ————————————————————
  gap> IntersectionForm(K);
  [ [ -1 ] ]
```

[The simplicial complex K in this second example is due to W. Kuehnel and T. F. Banchoff and is homeomorphic to the complex projective plane.]

3.7 Computing the second homotopy group of a space with infinite fundamental group

The following commands compute the second integral homology
$$H_2(\pi_1 W, \mathbb{Z}) = \mathbb{Z}$$
of the fundamental group $\pi_1 W$ of the complement W of the Hopf-Satoh surface.

```
———————————————————— Example ————————————————————
  gap> L:=HopfSatohSurface();
  Pure cubical complex of dimension 4.

  gap> W:=ContractedComplex(RegularCWComplex(PureComplexComplement(L)));
  Regular CW-complex of dimension 3

  gap> GW:=FundamentalGroup(W);;
  gap> IsAspherical(GW);
  Presentation is aspherical.
  true
  gap> R:=ResolutionAsphericalPresentation(GW);;
  gap> Homology(TensorWithIntegers(R),2);
  [ 0 ]
```

From Hopf's exact sequence
$$\pi_2 W \xrightarrow{h} H_2(W, \mathbb{Z}) \twoheadrightarrow H_2(\pi_1 W, \mathbb{Z}) \to 0$$
and the computation $H_2(W, \mathbb{Z}) = \mathbb{Z}^4$ we see that the image of the Hurewicz homomorphism is $\text{im}(h) = \mathbb{Z}^3$. The image of h is referred to as the subgroup of *spherical homology classes* and often denoted by $\Sigma^2 W$.

The following command computes the presentation of $\pi_1 W$ corresponding to the 2-skeleton W^2 and establishes that $W^2 = S^2 \vee S^2 \vee S^2 \vee (S^1 \times S^1)$ is a wedge of three spheres and a torus.

```
————————————— Example ——————————————
gap> F:=FundamentalGroupOfRegularCWComplex(W,"no simplification");
< fp group on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(F);
[ < identity ...>, f1^-1*f2^-1*f1*f2, < identity ...>, <identity ...> ]
```

The next command shows that the 3-dimensional space W has two 3-cells each of which is attached to the base-point of W with trivial boundary (up to homotopy in W^2). Hence $W = S^3 \vee S^3 \vee S^2 \vee S^2 \vee S^2 \vee (S^1 \times S^1)$.

```
————————————— Example ——————————————
gap> CriticalCells(W);
[ [ 3, 1 ], [ 3, 3148 ], [ 2, 6746 ], [ 2, 20510 ], [ 2, 33060 ],
  [ 2, 50919 ], [ 1, 29368 ], [ 1, 50822 ], [ 0, 21131 ] ]
gap> CriticalBoundaryCells(W,3,1);
[  ]
gap> CriticalBoundaryCells(W,3,3148);
[ -50919, 50919 ]
```

Therefore $\pi_1 W$ is the free abelian group on two generators, and $\pi_2 W$ is the free $\mathbb{Z}\pi_1 W$-module on three free generators.

Chapter 4

Three Manifolds

4.1 Dehn Surgery

The following example constructs, as a regular CW-complex, a closed orientable 3-manifold W obtained from the 3-sphere by drilling out a tubular neighbourhood of a trefoil knot and then gluing a solid torus to the boundary of the cavity via a homeomorphism corresponding to a Dehn surgery coefficient $p/q = 17/16$.

```
─────────────────────────── Example ───────────────────────────
  gap> ap:=ArcPresentation(PureCubicalKnot(3,1));;
  gap> p:=17;;q:=16;;
  gap> W:=ThreeManifoldViaDehnSurgery(ap,p,q);
  Regular CW-complex of dimension 3
```

The next commands show that this 3-manifold W has integral homology

$H_0(W, \mathbb{Z}) = \mathbb{Z}$, $H_1(W, \mathbb{Z}) = \mathbb{Z}_{16}$, $H_2(W, \mathbb{Z}) = 0$, $H_3(W, \mathbb{Z}) = \mathbb{Z}$

and that the fundamental group $\pi_1(W)$ is non-abelian.

```
─────────────────────────── Example ───────────────────────────
  gap> Homology(W,0);Homology(W,1);Homology(W,2);Homology(W,3);
  [ 0 ]
  [ 16 ]
  [  ]
  [ 0 ]

  gap> F:=FundamentalGroup(W);;
  gap> L:=LowIndexSubgroupsFpGroup(F,10);;
  gap> List(L,AbelianInvariants);
  [ [ 16 ], [ 3, 8 ], [ 3, 4 ], [ 2, 3 ], [ 16, 43 ], [ 8, 43, 43 ] ]
```

The following famous result of Lickorish and (independently) Wallace shows that Dehn surgery on knots leads to an interesting range of spaces.

THEOREM: *Every closed, orientable, connected 3-manifold can be obtained by surgery on a link in S^3. (Moreover, one can always perform the surgery with surgery coefficients ± 1 and with each individual component of the link unknotted.)*

4.2 Connected Sums

The following example constructs the connected sum $W = A\#B$ of two 3-manifolds, where A is obtained from a 5/1 Dehn surgery on the complement of the first prime knot on 11 crossings and B is obtained by a 5/1 Dehn surgery on the complement of the second prime knot on 11 crossings. The homology groups

$H_1(W,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_{594}, H_2(W,\mathbb{Z}) = 0, H_3(W,\mathbb{Z}) = \mathbb{Z}$

are computed.

```
——————————— Example ———————————
gap> ap1:=ArcPresentation(PureCubicalKnot(11,1));;
gap> A:=ThreeManifoldViaDehnSurgery(ap1,5,1);;
gap> ap2:=ArcPresentation(PureCubicalKnot(11,2));;
gap> B:=ThreeManifoldViaDehnSurgery(ap2,5,1);;
gap> W:=ConnectedSum(A,B);
Regular CW-complex of dimension 3

gap> Homology(W,1);
[ 2, 594 ]
gap> Homology(W,2);
[  ]
gap> Homology(W,3);
[ 0 ]
```

4.3 Dijkgraaf-Witten Invariant

Given a closed connected orientable 3-manifold W, a finite group G and a 3-cocycle $\alpha \in H^3(BG, U(1))$ Dijkgraaf and Witten define the complex number

$$ Z^{G,\alpha}(W) = \frac{1}{|G|}\sum_{\gamma\in {\rm Hom}(\pi_1W, G)} \langle \gamma^\ast[\alpha], [M]\rangle \ \in\ \mathbb{C} $$

where γ ranges over all group homomorphisms $\gamma\colon \pi_1 W \to G$. This complex number is an invariant of the homotopy type of W and is useful for distinguishing between certain homotopically distinct 3-manifolds.

A homology version of the Dijkgraaf-Witten invariant can be defined as the set of homology homomorphisms $$D_G(W) =\{ \gamma_\ast\colon H_3(W,\mathbb{Z}) \longrightarrow H_3(BG,\mathbb{Z}) \}_{\gamma\in {\rm Hom}(\pi_1W, G)}.$$ Since $H_3(W,\mathbb{Z}) \cong \mathbb{Z}$ we represent $D_G(W)$ by the set $D_G(W) = \{\gamma_\ast(1)\}_{\gamma\in \mathrm{Hom}(\pi_1W,G)}$ where 1 denotes one of the two possible generators of $H_3(W,\mathbb{Z})$.

For any coprime integers $p,q \geq 1$ the *lens space* $L(p,q)$ is obtained from the 3-sphere by drilling out a tubular neighbourhood of the trivial knot and then gluing a solid torus to the boundary of the cavity via a homeomorphism corresponding to a Dehn surgery coefficient p/q. Lens spaces have cyclic fundamental group $\pi_1(L(p,q)) = C_p$ and homology $H_1(L(p,q),\mathbb{Z}) \cong \mathbb{Z}_p, H_2(L(p,q),\mathbb{Z}) \cong 0,$ $H_3(L(p,q),\mathbb{Z}) \cong \mathbb{Z}$. It was proved by J.H.C. Whitehead that two lens spaces $L(p,q)$ and $L(p',q')$ are homotopy equivalent if and only if $p = p'$ and $qq' \equiv \pm n^2$ mod p for some integer n.

The following session constructs the two lens spaces $L(5,1)$ and $L(5,2)$. The homology version of the Dijkgraaf-Witten invariant is used with $G = C_5$ to demonstrate that the two lens spaces are not homotopy equivalent.

```
——————————— Example ———————————
gap> ap:=[[2,1],[2,1]];; #Arc presentation for the trivial knot
```

```
gap> L51:=ThreeManifoldViaDehnSurgery(ap,5,1);;
gap> D:=DijkgraafWittenInvariant(L51,CyclicGroup(5));
[ g1^4, g1^4, g1, g1, id ]

gap> L52:=ThreeManifoldViaDehnSurgery(ap,5,2);;
gap> D:=DijkgraafWittenInvariant(L52,CyclicGroup(5));
[ g1^3, g1^3, g1^2, g1^2, id ]
```

A theorem of Fermat and Euler states that if a prime p is congruent to 3 modulo 4, then for any q exactly one of $\pm q$ is a quadratic residue mod p. For all other primes p either both or neither of $\pm q$ is a quadratic residue mod p. Thus for fixed $p \equiv 3 \mod 4$ the lens spaces $L(p,q)$ form a single homotopy class. There are precisely two homotopy classes of lens spaces for other p.

The following commands confirm that $L(13,1) \not\simeq L(13,2)$.

```
                             Example
gap> L13_1:=ThreeManifoldViaDehnSurgery([[1,2],[1,2]],13,1);;
gap> DijkgraafWittenInvariant(L13_1,CyclicGroup(13));
[ g1^12, g1^12, g1^10, g1^10, g1^9, g1^9, g1^4, g1^4, g1^3, g1^3, g1, g1, id ]
gap> L13_2:=ThreeManifoldViaDehnSurgery([[1,2],[1,2]],13,2);;
gap> DijkgraafWittenInvariant(L13_2,CyclicGroup(13));
[ g1^11, g1^11, g1^8, g1^8, g1^7, g1^7, g1^6, g1^6, g1^5, g1^5, g1^2, g1^2,
  id ]
```

4.4   Cohomology rings

The following commands construct the multiplication table (with respect to some basis) for the cohomology rings $H^*(L(13,1),\mathbb{Z}_{13})$ and $H^*(L(13,2),\mathbb{Z}_{13})$. These rings are isomorphic and so fail to distinguish between the homotopy types of the lens spaces $L(13,1)$ and $L(13,2)$.

```
                             Example
gap> L13_1:=ThreeManifoldViaDehnSurgery([[1,2],[1,2]],13,1);;
gap> L13_2:=ThreeManifoldViaDehnSurgery([[1,2],[1,2]],13,2);;
gap> L13_1:=BarycentricSubdivision(L13_1);;
gap> L13_2:=BarycentricSubdivision(L13_2);;
gap> A13_1:=CohomologyRing(L13_1,13);;
gap> A13_2:=CohomologyRing(L13_2,13);;
gap> M13_1:=List([1..4],i->[]);;
gap> B13_1:=CanonicalBasis(A13_1);;
gap> M13_2:=List([1..4],i->[]);;
gap> B13_2:=CanonicalBasis(A13_2);;
gap> for i in [1..4] do
> for j in [1..4] do
> M13_1[i][j]:=B13_1[i]*B13_1[j];
> od;od;
gap> for i in [1..4] do
> for j in [1..4] do
> M13_2[i][j]:=B13_2[i]*B13_2[j];
> od;od;
gap> Display(M13_1);
[ [            v.1,            v.2,            v.3,            v.4 ],
```

```
    [           v.2,          0*v.1,   (Z(13)^6)*v.4,          0*v.1 ],
    [           v.3,   (Z(13)^6)*v.4,          0*v.1,          0*v.1 ],
    [           v.4,          0*v.1,          0*v.1,          0*v.1 ] ]
gap> Display(M13_2);
[ [           v.1,            v.2,            v.3,            v.4 ],
  [           v.2,          0*v.1,     (Z(13))*v.4,          0*v.1 ],
  [           v.3,     (Z(13))*v.4,          0*v.1,          0*v.1 ],
  [           v.4,          0*v.1,          0*v.1,          0*v.1 ] ]
```

4.5   Linking Form

Given a closed connected ORIENTED 3-manifold W let $\tau H_1(W,\mathbb{Z})$ denote the torsion subgroup of the first integral homology. The *linking form* is a bilinear mapping

$Lk_W: \tau H_1(W,\mathbb{Z}) \times \tau H_1(W,\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$.

To construct this form note that we have a Poincare duality isomorphism

$\rho: H^2(W,\mathbb{Z}) \xrightarrow{\cong} H_1(W,\mathbb{Z}), z \mapsto z \cap [W]$

involving the cap product with the fundamental class $[W] \in H^3(W,\mathbb{Z})$. That is, $[M]$ is the generator of $H^3(W,\mathbb{Z}) \cong \mathbb{Z}$ determining the orientation. The short exact sequence $\mathbb{Z} \rightarrowtail \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ gives rise to a cohomology exact sequence

$\rightarrow H^1(W,\mathbb{Q}) \rightarrow H^1(W,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H^2(W,\mathbb{Z}) \rightarrow H^2(W,\mathbb{Q}) \rightarrow$

from which we obtain the isomorphism $\beta: \tau H^1(W,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \tau H^2(W,\mathbb{Z})$. The linking form Lk_W can be defined as the composite

$Lk_W: \tau H_1(W,\mathbb{Z}) \times \tau H_1(W,\mathbb{Z}) \xrightarrow{1 \times \rho^{-1}} \tau H_1(W,\mathbb{Z}) \times \tau H^2(W,\mathbb{Z}) \xrightarrow{1 \times \beta^{-1}} \tau H_1(W,\mathbb{Z}) \times \tau H^1(W,\mathbb{Q}/\mathbb{Z}) \xrightarrow{ev} \mathbb{Q}/\mathbb{Z}$

where $ev(x,\alpha)$ evaluates a 1-cocycle α on a 1-cycle x.

The linking form can be used to define the set

$I^O(W) = \{Lk_W(g,g) : g \in \tau H_1(W,\mathbb{Z})\}$

which is an oriented-homotopy invariant of W. Letting W^+ and W^- denote the two possible orientations on the manifold, the set

$I(W) = \{I^O(W^+), I^O(W^-)\}$

is a homotopy invariant of W which in this manual we refer to as the *linking form homotopy invariant*.

The following commands compute the linking form homotopy invariant for the lens spaces $L(13,q)$ with $1 \le q \le 12$. This invariant distinguishes between the two homotopy types that arise.

```
────────────────────── Example ──────────────────────
gap> LensSpaces:=[];;
gap> for q in [1..12] do
> Add(LensSpaces,ThreeManifoldViaDehnSurgery([[1,2],[1,2]],13,q));
> od;
gap> Display(List(LensSpaces,LinkingFormHomotopyInvariant));;
[ [ [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ],
      [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ] ],

  [ [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ], [ 0, 2/13,
        2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ] ],
```

```
   [ [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ],
       [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ] ],

   [ [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ],
       [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ] ],

   [ [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ],
       [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ] ],

   [ [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ],
       [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ] ],

   [ [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ],
       [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ] ],

   [ [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ],
       [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ] ],

   [ [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ],
       [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ] ],

   [ [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ],
       [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ] ],

   [ [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ],
       [ 0, 2/13, 2/13, 5/13, 5/13, 6/13, 6/13, 7/13, 7/13, 8/13, 8/13, 11/13, 11/13 ] ],

   [ [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ],
       [ 0, 1/13, 1/13, 3/13, 3/13, 4/13, 4/13, 9/13, 9/13, 10/13, 10/13, 12/13, 12/13 ] ] ]
```

4.6   Determining the homeomorphism type of a lens space

In 1935 K. Reidemeister [Rei35] classified lens spaces up to orientation preserving PL-homeomorphism. This was generalized by E. Moise [Moi52] in 1952 to a classification up to homeomorphism – his method requred the proof of the Hauptvermutung for 3-dimensional manifolds. In 1960, following a suggestion of R. Fox, a proof was given by E.J. Brody [Bro60] that avoided the need for the Hauptvermutung. Reidemeister's method, using what is know termed *Reidermeister torsion*, and Brody's method, using tubular neighbourhoods of 1-cycles, both require identifying a suitable "preferred" generator of $H_1(L(p,q),\mathbb{Z})$. In 2003 J. Przytycki and A. Yasukhara [PY03] provided an alternative method for classifying lens spaces, which uses the linking form and again requires the identification of a "preferred" generator of $H_1(L(p,q),\mathbb{Z})$.

Przytycki and Yasukhara proved the following.

THEOREM. *Let $\rho\colon S^3 \to L(p,q)$ be the p-fold cyclic cover and K a knot in $L(p,q)$ that represents a generator of $H_1(L(p,q),\mathbb{Z})$. If $\rho^{-1}(K)$ is the trivial knot, then $Lk_{L(p,q)}([K],[K]) = q/p$ or $= \overline{q}/p \in \mathbb{Q}/\mathbb{Z}$ where $q\overline{q} \equiv 1 \bmod p$.*

The ingredients of this theorem can be applied in HAP, but at present only to small examples of lens spaces. The obstruction to handling large examples is that the current default method for

computing the linking form involves barycentric subdivision to produce a simplicial complex from a regular CW-complex, and then a homotopy equivalence from this typically large simplicial complex to a smaller non-regular CW-complex. However, for homeomorphism invariants that are not homotopy invariants there is a need to avoid homotopy equivalences. In the current version of HAP this means that in order to obtain delicate homeomorphism invariants we have to perform homology computations on typically large simplicial complexes. In a future version of HAP we hope to avoid the obstruction by implementing cup products, cap products and linking forms entirely within the category of regular CW-complexes.

The following commands construct a small lens space $L = L(p,q)$ with unknown values of p,q. Subsequent commands will determine the homeomorphism type of L.

```
─────────────────── Example ───────────────────
gap> p:=Random([2,3,5,7,11,13,17]);;
gap> q:=Random([1..p-1]);;
gap> L:=ThreeManifoldViaDehnSurgery([[1,2],[1,2]],p,q);
Regular CW-complex of dimension 3
```

We can readily determine the value of $p = 11$ by calculating the order of $\pi_1(L)$.

```
─────────────────── Example ───────────────────
gap> F:=FundamentalGroupWithPathReps(L);;
gap> StructureDescription(F);
"C11"
```

The next commands take the default edge path $\gamma\colon S^1 \to L$ representing a generator of the cyclic group $\pi_1(L)$ and lift it to an edge path $\tilde{\gamma}\colon S^1 \to \tilde{L}$.

```
─────────────────── Example ───────────────────
gap> U:=UniversalCover(L);;
gap> G:=U!.group;;
gap> p:=EquivariantCWComplexToRegularCWMap(U,Group(One(G)));;
gap> U:=Source(p);;
gap> gamma:=[];;
gap> gamma[2]:=F!.loops[1];;
gap> gamma[2]:=List(gamma[2],AbsInt);;
gap> gamma[1]:=List(gamma[2],k->L!.boundaries[2][k]);;
gap> gamma[1]:=SSortedList(Flat(gamma[1]));;
gap>
gap> gammatilde:=[[],[],[],[]];;
gap> for k in [1..U!.nrCells(0)] do
> if p!.mapping(0,k) in gamma[1] then Add(gammatilde[1],k); fi;
> od;
gap> for k in [1..U!.nrCells(1)] do
> if p!.mapping(1,k) in gamma[2] then Add(gammatilde[2],k); fi;
> od;
gap> gammatilde:=CWSubcomplexToRegularCWMap([U,gammatilde]);
Map of regular CW-complexes
```

The next commands check that the path $\tilde{\gamma}$ is unknotted in $\tilde{L} \cong S^3$ by checking that $\pi_1(\tilde{L} \setminus \mathrm{image}(\tilde{\gamma}))$ is infinite cyclic.

```
gap> C:=RegularCWComplexComplement(gammatilde);
Regular CW-complex of dimension 3

gap> G:=FundamentalGroup(C);
<fp group of size infinity on the generators [ f2 ]>
```

Since $\tilde{\gamma}$ is unkotted the cycle γ represents the preferred generator $[\gamma] \in H_1(L, \mathbb{Z})$. The next commands compute $Lk_L([\gamma], [\gamma]) = 7/11$.

```
gap> LinkingFormHomeomorphismInvariant(L);
[ 7/11 ]
```

The classification of Moise/Brody states that $L(p,q) \cong L(p,q')$ if and only if $qq' \equiv \pm 1 \bmod p$. Hence the lens space L has the homeomorphism type
$$L \cong L(11,7) \cong L(11,8) \cong L(11,4) \cong L(11,3).$$

4.7 Surgeries on distinct knots can yield homeomorphic manifolds

The lens space $L(5,1)$ is a quotient of the 3-sphere S^3 by a certain action of the cyclic group C_5. It can be realized by a $p/q = 5/1$ Dehn filling of the complement of the trivial knot. It can also be realized by Dehn fillings of other knots. To see this, the following commands compute the manifold W obtained from a $p/q = 1/5$ Dehn filling of the complement of the trefoil and show that W at least has the same integral homology and same fundamental group as $L(5,1)$.

```
gap> ap:=ArcPresentation(PureCubicalKnot(3,1));;
gap> W:=ThreeManifoldViaDehnSurgery(ap,1,5);;
gap> Homology(W,1);
[ 5 ]
gap> Homology(W,2);
[  ]
gap> Homology(W,3);
[ 0 ]

gap> F:=FundamentalGroup(W);;
gap> StructureDescription(F);
"C5"
```

The next commands construct the universal cover $\widetilde{W}$ and show that it has the same homology as S^3 and trivivial fundamental group $\pi_1(\widetilde{W}) = 0$.

```
gap> U:=UniversalCover(W);;
gap> G:=U!.group;;
gap> Wtilde:=EquivariantCWComplexToRegularCWComplex(U,Group(One(G)));
Regular CW-complex of dimension 3
```

```
gap> Homology(Wtilde,1);
[  ]
gap> Homology(Wtilde,2);
[  ]
gap> Homology(Wtilde,3);
[ 0 ]

gap> F:=FundamentalGroup(Wtilde);
<fp group on the generators [  ]>
```

By construction the space $\widetilde{W}$ is a manifold. Had we not known how the regular CW-complex $\widetilde{W}$ had been constructed then we could prove that it is a closed 3-manifold by creating its barycentric subdivision $K = sd\widetilde{W}$, which is homeomorphic to $\widetilde{W}$, and verifying that the link of each vertex in the simplicial complex $sd\widetilde{W}$ is a 2-sphere. The following command carries out this proof.

```
————————————————————— Example —————————————————————
gap> IsClosedManifold(Wtilde);

true
```

The Poincare conjecture (now proven) implies that $\widetilde{W}$ is homeomorphic to S^3. Hence $W = S^3/C_5$ is a quotient of the 3-sphere by an action of C_5 and is hence a lens space $L(5,q)$ for some q.

The next commands determine that W is homeomorphic to $L(5,4) \cong L(5,1)$.

```
————————————————————— Example —————————————————————
gap> Lk:=LinkingFormHomeomorphismInvariant(W);
[ 0, 4/5 ]
```

Moser [Mos71] gives a precise decription of the lens spaces arising from surgery on the trefoil knot and more generally from surgery on torus knots. Greene [Gre13] determines the lens spaces that arise by integer Dehn surgery along a knot in the three-sphere

4.8 Finite fundamental groups of 3-manifolds

Lens spaces are examples of 3-manifolds with finite fundamental groups. The complete list of finite groups G arising as fundamental groups of closed connected 3-manifolds is recalled in 7.11 where one method for computing their cohomology rings is presented. Their cohomology could also be computed from explicit 3-manifolds W with $\pi_1 W = G$. For instance, the following commands realize a closed connected 3-manifold W with $\pi_1 W = C_{11} \times SL_2(\mathbb{Z}_5)$.

```
————————————————————— Example —————————————————————
gap> ap:=ArcPresentation(PureCubicalKnot(3,1));;
gap> W:=ThreeManifoldViaDehnSurgery(ap,1,11);;
gap> F:=FundamentalGroup(W);;
gap> Order(F);
1320
gap> AbelianInvariants(F);
[ 11 ]
gap> StructureDescription(F);
```

```
  "C11 x SL(2,5)"
```

Hence the group $G = C_{11} \times SL_2(\mathbb{Z}_5)$ of order 1320 acts freely on the 3-sphere $\widetilde{W}$. It thus has periodic cohomology with

$$H_n(G, \mathbb{Z}) = \begin{cases} \mathbb{Z}_{11} & n \equiv 1 \bmod 4 \\ 0 & n \equiv 2 \bmod 4 \\ \mathbb{Z}_{1320} & n \equiv 3 \bmod 4 \\ \nvdash & n \equiv 0 \bmod 4 \end{cases}$$

for $n > 0$.

4.9 Poincare's cube manifolds

In his seminal paper on "Analysis situs", published in 1895, Poincare constructed a series of closed 3-manifolds which played an important role in the development of his theory. A good account of these manifolds is given in the online Manifold Atlas Project (MAP). Most of his examples are constructed by identifications on the faces of a (solid) cube. The function `PoincareCubeCWComplex()` can be used to construct any 3-dimensional CW-complex arising from a cube by identifying the six faces pairwise; the vertices and faces of the cube are numbered as follows

and barycentric subdivision is used to ensure that the quotient is represented as a regular CW-complex.

Examples 3 and 4 from Poincare's paper, described in the following figures taken from MAP,

are constructed in the following example. Both are checked to be orientable manifolds, and are shown to have different homology. (Note that the second example in Poincare's paper is not a manifold – the links of some of its vertices are not homeomorphic to a 2-sphere.)

─────────────────────────────── Example ───────────────────────────────
```
gap> A:=1;;C:=2;;D:=3;;B:=4;;
gap> Ap:=5;;Cp:=6;;Dp:=7;;Bp:=8;;

gap> L:=[[A,B,D,C],[Bp,Dp,Cp,Ap]];;
gap> M:=[[A,B,Bp,Ap],[Cp,C,D,Dp]];;
gap> N:=[[A,C,Cp,Ap],[D,Dp,Bp,B]];;
gap> Ex3:=PoincareCubeCWComplex(L,M,N);
Regular CW-complex of dimension 3

gap> IsClosedManifold(Ex3);

true

gap> L:=[[A,B,D,C],[Bp,Dp,Cp,Ap]];;
gap> M:=[[A,B,Bp,Ap],[C,D,Dp,Cp]];;
gap> N:=[[A,C,Cp,Ap],[B,D,Dp,Bp]];;
gap> Ex4:=PoincareCubeCWComplex(L,M,N);
Regular CW-complex of dimension 3

gap> IsClosedManifold(Ex4);
true

gap> List([0..3],k->Homology(Ex3,k));
```

```
[ [ 0 ], [ 2, 2 ], [ ], [ 0 ] ]
gap> List([0..3],k->Homology(Ex4,k));
[ [ 0 ], [ 2, 0 ], [ 0 ], [ 0 ] ]
```

4.10   There are (at least) 25 distinct cube manifolds

The function `PoincareCubeCWComplex(A,G)` can also be applied to two inputs where A is a pairing of the six faces such as $A = [[1,2],[3,4],[5,6]]$ and G is a list of three elements of the dihedral group of order 8 such as $G = [(2,4),(2,4),(2,4)*(1,3)]$. The dihedral elements specify how each pair of faces are glued together. With these inputs it is easy to iterate over all possible values of A and G in order to construct all possible 3-manifolds arising from the pairwise identification of faces of a cube. Distinct values of A and G can of course yield homeomorphic spaces. To ensure that each possible cube manifold is constructed, at least once, up to homeomorphism it suffices to consider
  $A = [[1,2],[3,4],[5,6]], A = [[1,2],[3,5],[4,6]], A = [[1,4],[2,6],[3,5]]$
  and all G in $D_8 \times D_8 \times D_8$.

The following commands iterate through these $3 \times 8^3 = 1536$ pairs (A,G) and show that in precisely 163 cases (just over 10% of cases) the quotient CW-complex is a 3-manifold.

```
───────────────── Example ─────────────────
 A1:= [ [1,2], [3,4], [5,6] ];
 A2:=[ [1,2], [3,5], [4,6] ];
 A3:=[ [1,4], [2,6], [3,5] ];
 D8:=DihedralGroup(IsPermGroup,8);

 Manifolds:=[];
 for A in [A1,A2,A3] do
 for x in D8 do
 for y in D8 do
 for z in D8 do
 G:=[x,y,z];
 F:=PoincareCubeCWComplex(A,G);
 b:=IsClosedManifold(F);
 if b=true then Add(Manifolds,F); fi;
 od;od;od;od;

 Size(Manifolds);
 163
```

The following additional commands use integral homology and low index subgroups of fundamental groups to establish that the 163 cubes manifolds represent at least 25 distinct homotopy equivalence classes of manifolds. One homotopy class is represented by up to 40 of the manifolds, and at least four of the homotopy classes are each represented by a single manifold..

```
───────────────── Example ─────────────────
 invariant1:=function(m);
 return List([1..3],k->Homology(m,k));
 end;;

 C:=Classify(M,invariant1);;
```

```
invariant2:=function(m)
local L;
L:=FundamentalGroup(m);
if GeneratorsOfGroup(L)= [] then return [];fi;
L:=LowIndexSubgroupsFpGroup(L,10);
L:=List(L,AbelianInvariants);
L:=SortedList(L);
return L;
end;;

D:=RefineClassification(C,invariant2);
List(D,Size);
[ 40, 2, 10, 15, 8, 6, 2, 6, 2, 5, 7, 1, 4, 11, 7, 7, 10, 4, 4, 2, 1, 3, 1,
  1, 4 ]
```

The next commands construct a list of 18 orientable cube manifolds and a list of 7 non-orientable cube manifolds.

```
───────────── Example ─────────────
gap> Manifolds:=List(Manifolds,x->x[1]);;
gap> OrientablManifolds:=Filtered(Manifolds,m->Homology(m,3)=[0]);;
gap> NonOrientablManifolds:=Filtered(Manifolds,m->Homology(m,3)=[]);;
gap> Length(OrientablManifolds);
18
gap> Length(NonOrientablManifolds);
7
```

The next commands show that the 7 non-orientable cube manifolds all have infinite fundamental groups.

```
───────────── Example ─────────────
gap> List(NonOrientablManifolds,m->IsFinite(FundamentalGroup(m)));
[ false, false, false, false, false, false, false ]
```

The final commands show that (at least) 9 of the orientable manifolds have finite fundamental groups and list the isomorphism types of these finite groups. Note that it is now known that any 3-manifold with finite fundamental group is spherical (i.e. is a quotient of the 3-sphere).

```
───────────── Example ─────────────
gap> List(OrientablManifolds{[4,8,10,11,12,13,15,16,18]},m->
          IsFinite(FundamentalGroup(m)));
[ true, true, true, true, true, true, true, true, true ]

gap> List(OrientablManifolds{[4,8,10,11,12,13,15,16,18]},m->
          StructureDescription(FundamentalGroup(m)));
[ "Q8", "C2", "C4", "C3 : C4", "C12", "C8", "C14", "C6", "1" ]
```

Chapter 5

Topological data analysis

5.1 Persistent homology

Pairwise distances between 74 points from some metric space have been recorded and stored in a 74×74 matrix D. The following commands load the matrix, construct a filtration of length 100 on the first two dimensions of the assotiated clique complex (also known as the *Rips Complex*), and display the resulting degree 0 persistent homology as a barcode. A single bar with label n denotes n bars with common starting point and common end point.

```
——————————————— Example ———————————————
gap> file:=HapFile("data253a.txt");;
gap> Read(file);

gap> G:=SymmetricMatrixToFilteredGraph(D,100);
Filtered graph on 74 vertices.

gap> K:=FilteredRegularCWComplex(CliqueComplex(G,2));
Filtered regular CW-complex of dimension 2

gap> P:=PersistentBettiNumbers(K,0);;
gap> BarCodeCompactDisplay(P);
```

The next commands display the resulting degree 1 persistent homology as a barcode.

```
——————————————— Example ———————————————
gap> P:=PersistentBettiNumbers(K,1);;
gap> BarCodeCompactDisplay(P);
```

The following command displays the 1 skeleton of the simplicial complex arizing as the 65-th term in the filtration on the clique complex.

```
——————————————— Example ———————————————
gap> Y:=FiltrationTerm(K,65);
Regular CW-complex of dimension 1

gap> Display(HomotopyGraph(Y));
```

These computations suuggest that the dataset contains two persistent path components (or clusters), and that each path component is in some sense periodic. The final command displays one possible representation of the data as points on two circles.

5.1.1 Background to the data

Each point in the dataset was an image consisting of 732×761 pixels. This point was regarded as a vector in $\mathbb{R}^{732 \times 761}$ and the matrix D was constructed using the Euclidean metric. The images were the following:

5.2 Mapper clustering

The following example reads in a set S of vectors of rational numbers. It uses the Euclidean distance $d(u, v)$ between vectors. It fixes some vector $u_0 \in S$ and uses the associated function $f : D \rightarrow [0, b] \subset \mathbb{R}, v \mapsto d(u_0, v)$. In addition, it uses an open cover of the interval $[0, b]$ consisting of 100 uniformly distributed overlapping open subintervals of radius $r = 29$. It also uses a simple clustering algorithm implemented in the function `cluster`.

These ingredients are input into the Mapper clustering procedure to produce a simplicial complex M which is intended to be a representation of the data. The complex M is 1-dimensional and the final command uses GraphViz software to visualize the graph. The nodes of this simplicial complex are "buckets" containing data points. A data point may reside in several buckets. The number of points in the bucket determines the size of the node. Two nodes are connected by an edge when their end-point nodes contain common data points.

```
———————————————— Example ————————————————
  gap> file:=HapFile("data134.txt");;
  gap> Read(file);
  gap> dx:=EuclideanApproximatedMetric;;
  gap> dz:=EuclideanApproximatedMetric;;
  gap> L:=List(S,x->Maximum(List(S,y->dx(x,y))));;
  gap> n:=Position(L,Minimum(L));;
  gap> f:=function(x); return [dx(S[n],x)]; end;;
  gap> P:=30*[0..100];; P:=List(P, i->[i]);;
  gap> r:=29;;
  gap> epsilon:=75;;
  gap>  cluster:=function(S)
  >   local Y, P, C;
  >   if Length(S)=0 then return S; fi;
  >   Y:=VectorsToOneSkeleton(S,epsilon,dx);
  >   P:=PiZero(Y);
  >   C:=Classify([1..Length(S)],P[2]);
  >   return List(C,x->S{x});
  >  end;;
  gap> M:=Mapper(S,dx,f,dz,P,r,cluster);
  Simplicial complex of dimension 1.

  gap> Display(GraphOfSimplicialComplex(M));
```

5.2.1 Background to the data

The datacloud *S* consists of the 400 points in the plane shown in the following picture.

5.3 Digital image analysis

The following example reads in a digital image as a filtered pure cubical complexex. The filtration is obtained by thresholding at a sequence of uniformly spaced values on the greyscale range. The persistent homology of this filtered complex is calculated in degrees 0 and 1 and displayed as two barcodes.

```
———————————— Example ————————————
gap> file:=HapFile("image1.3.2.png");;
gap> F:=ReadImageAsFilteredPureCubicalComplex(file,20);
Filtered pure cubical complex of dimension 2.
gap> P:=PersistentBettiNumbers(F,0);;
gap> BarCodeCompactDisplay(P);
```

```
———————————— Example ————————————
gap> P:=PersistentBettiNumbers(F,1);;
gap> BarCodeCompactDisplay(P);
```

The 20 persistent bars in the degree 0 barcode suggest that the image has 20 objects. The degree 1 barcode suggests that 14 (or possibly 17) of these objects have holes in them.

5.3.1 Background to the data

The following image was used in the example.

5.4 Random simplicial complexes

For a positive integer n and probability p we denote by $Y(n,p)$ the *Linial-Meshulam random simplicial 2-complex*. Its 1-skeleton is the complete graph on n vertices; each possible 2-simplex is included independently with probability p.

The following commands first compute the number h_i of non-trivial cyclic summands in $H_i(Y(100,p),\mathbb{Z})$ for a range of probabilities p and $i = 1,2$ and then produce a plot of h_i versus p. The plot for h_1 is red and the plot for h_2 is blue. A plot for the Euler characteristic $1 - h_1 + h_2$ is shown in green.

```
———————————— Example ————————————
gap> L:=[];;M:=[];;
gap> for p in [1..100] do
> K:=RegularCWComplex(RandomSimplicialTwoComplex(100,p/1000));;
> h1:=Length(Homology(K,1));;
> h2:=Length(Homology(K,2));;
> Add(L, [1.0*(p/1000),h1,"red"]);
> Add(L, [1.0*(p/1000),h2,"blue"]);
> Add(M, [1.0*(p/1000),1-h1+h2,"green"]);
> od;
```

```
gap> ScatterPlot(L);
gap> ScatterPlot(M);
```

From this plot it seems that there is a *phase change threshold* at around $p = 0.025$. An inspection of the first homology groups $H_1(Y(100, p), \mathbb{Z})$ shows that in most cases there is no torsion. However, around the threshold some of the complexes do have torsion in their first homology.

Similar commands for $Y(75, p)$ suggest a phase transition at around $p = 0.035$ in this case. The following commands compute $H_1(Y(75, p), \mathbb{Z})$ for 900 random 2-complexes with p in a small interval around 0.035 and, in each case where there is torsion, the torsion coefficients are stored in a list. The final command prints these lists – all but one of which are of length 1. For example, there is one 2-dimensional simplicial complex on 75 vertices whose first homology contains the summand $\mathbb{Z}_{10787966187051680066516118257882128}$. The largest prime factor is $805552359079941450096902063$ occuring in the summand $\mathbb{Z}_{2598377606162872942310817669788555}$.

―――――――――――――――――――――――― Example ――――――――――――――――――――――――
```
gap> torsion:=function(n,p)
> local H, Y;
> Y:=RegularCWComplex(RandomSimplicialTwoComplex(n,p));
> H:=Homology(Y,1);
> H:=Filtered(H,x->not x=0);
> return H;
> end;
function( n, p ) ... end


gap> L:=[];;for n in [73000..73900] do
> t:=torsion(75,n/2000000);
> if not t=[] then Add(L,t); fi;
> od;

gap> Display(L);
[ [                                         2 ],
  [                                        26 ],
  [      2598377606162872942310817669788555 ],
  [                                         2 ],
  [                                         3 ],
  [                                         2 ],
  [           2761642698060127444812143568 ],
  [         262635528101097466377627338196 ],
  [                                         2 ],
  [                                         3 ],
  [         331123827512648948194307858350 ],
  [                                        16 ],
  [                                         4 ],
  [                                         3 ],
  [                                         2 ],
  [                                         3 ],
  [                                         2 ],
  [        852349499991838889677631005909977 ],
  [                                         2 ],
  [        246441961307858211078977186622022 ],
```

```
[                                            2,                              2 ],
[                                            2 ],
[            4166416628890256454492982468 ],
[            4158277300187503916878697 0816 ],
[                                            2 ],
[             75889883165411088431747730 ],
[            3352347409163655479230 5315165 ],
[  10787966187051680066516118 2578823128 ],
[             5588265814409119568341729980 ],
[                                            2 ],
[             5001457249224115878015053458 ],
[                                           10 ],
[                                           12 ],
[                                            2 ],
[                                            2 ],
[                                            3 ],
[             7757870243425246987971789322 ],
[            8164648856993269673396613497412 ],
[                                            2 ] ]
```

Chapter 6

Group theoretic computations

6.1 Third homotopy group of a supsension of an Eilenberg-MacLane space

The following example uses the nonabelian tensor square of groups to compute the third homotopy group

$$\pi_3(S(K(G,1))) = \mathbb{Z}^{30}$$

of the suspension of the Eigenberg-MacLane space $K(G,1)$ for G the free nilpotent group of class 2 on four generators.

```
───────────────────────── Example ─────────────────────────
gap> F:=FreeGroup(4);;G:=NilpotentQuotient(F,2);;
gap> ThirdHomotopyGroupOfSuspensionB(G);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

6.2 Representations of knot quandles

The following example constructs the finitely presented quandles associated to the granny knot and square knot, and then computes the number of quandle homomorphisms from these two finitely prresented quandles to the 17-th quandle in HAP's library of connected quandles of order 24. The number of homomorphisms differs between the two cases. The computation therefore establishes that the complement in $\mathbb{R}^3$ of the granny knot is not homeomorphic to the complement of the square knot.

```
───────────────────────── Example ─────────────────────────
gap> Q:=ConnectedQuandle(24,17,"import");;
gap> K:=PureCubicalKnot(3,1);;
gap> L:=ReflectedCubicalKnot(K);;
gap> square:=KnotSum(K,L);;
gap> granny:=KnotSum(K,K);;
gap> gcsquare:=GaussCodeOfPureCubicalKnot(square);;
gap> gcgranny:=GaussCodeOfPureCubicalKnot(granny);;
gap> Qsquare:=PresentationKnotQuandle(gcsquare);;
gap> Qgranny:=PresentationKnotQuandle(gcgranny);;
gap> NumberOfHomomorphisms(Qsquare,Q);
408
```

```
gap> NumberOfHomomorphisms(Qgranny,Q);
24
```

The following commands compute a knot quandle directly from a pdf file containing the following hand-drawn image of the knot.

```
——————————————————————— Example ———————————————————————
gap>  gc:=ReadLinkImageAsGaussCode("myknot.pdf");
[ [ [ -2, 4, -1, 3, -3, 2, -4, 1 ] ], [ -1, -1, 1, -1 ] ]
gap> Q:=PresentationKnotQuandle(gc);
Quandle presentation of 4 generators and 4 relators.
```

6.3   Identifying knots

Low index subgrops of the knot group can be used to identify knots with few crossings. For instance, the following commands read in the following image of a knot and identify it as a sum of two trefoils. The commands determine the prime components only up to reflection, and so they don't distinguish between the granny and square knots.

```
——————————————————————— Example ———————————————————————
gap> gc:=ReadLinkImageAsGaussCode("myknot2.png");
[ [ [ -4, 7, -5, 4, -7, 5, -3, 6, -2, 3, 8, -8, -6, 2, 1, -1 ] ],
  [ 1, -1, -1, -1, -1, -1, -1, 1 ] ]
gap> IdentifyKnot(gc);;
PrimeKnot(3,1) + PrimeKnot(3,1)    modulo reflections of components.
```

6.4   Aspherical 2-complexes

The following example uses Polymake's linear programming routines to establish that the 2-complex associated to the group presentation $< x, y, z : xyx = yxy, yzy = zyz, xzx = zxz >$ is aspherical (that is, has contractible universal cover). The presentation is Tietze equivalent to the presentation used in the computer code, and the associated 2-complexes are thus homotopy equivalent.

```
——————————————————————— Example ———————————————————————
gap> F:=FreeGroup(6);;
gap> x:=F.1;;y:=F.2;;z:=F.3;;a:=F.4;;b:=F.5;;c:=F.6;;
gap> rels:=[a^-1*x*y, b^-1*y*z, c^-1*z*x, a*x*(y*a)^-1,
>     b*y*(z*b)^-1, c*z*(x*c)^-1];;
gap> Print(IsAspherical(F,rels),"\n");
Presentation is aspherical.

true
```

6.5 Bogomolov multiplier

The Bogomolov multiplier of a group is an isoclinism invariant. Using this property, the following example shows that there are precisely three groups of order 243 with non-trivial Bogomolov multiplier. The groups in question are numbered 28, 29 and 30 in GAP's library of small groups of order 243.

```
———————————————— Example ————————————————
  gap> L:=AllSmallGroups(3^5);;
  gap> C:=IsoclinismClasses(L);;
  gap> for c in C do
  > if Length(BogomolovMultiplier(c[1]))>0 then
  > Print(List(c,g->IdGroup(g)),"\n\n\n"); fi;
  > od;
  [ [ 243, 28 ], [ 243, 29 ], [ 243, 30 ] ]
```

6.6 Second group cohomology and group extensions

Any group extension $N \rightarrowtail E \twoheadrightarrow G$ gives rise to:

- an outer action $\alpha: G \to Out(G)$ of G on N.

- an action $G \to Aut(Z(N))$ of G on the centre of N, uniquely induced by the outer action α and the canonical action of $Out(N)$ on $Z(N)$.

- a 2-cocycle $f: G \times G \to Z(N)$ with values in the G-module $A = Z(N)$.

Any outer homomorphism $\alpha: G \to Out(N)$ gives rise to a cohomology class k in $H^3(G, Z(N))$. It was shown by Eilenberg and MacLane that the class k is trivial if and only if the outer action α arises from some group extension $N \rightarrowtail E \twoheadrightarrow G$. If k is trivial then there is a bijection between the second cohomology group $H^2(G, Z(N))$ and Yoneda equivalence classes of extensions of G by N that are compatible with α.

FIRST EXAMPLE.

Consider the group $H = SmallGroup(64, 134)$. Consider the normal subgroup $N = NormalSubgroups(G)[15]$ and quotient group $G = H/N$. We have $N = C_2 \times D_4$, $A = Z(N) = C_2 \times C_2$ and $G = C_2 \times C_2$.

Suppose we wish to classify all extensions $C_2 \times D_4 \rightarrowtail E \twoheadrightarrow C_2 \times C_2$ that induce the given outer action of G on N. The following commands show that, up to Yoneda equivalence, there are two such extensions.

```
———————————————— Example ————————————————
  gap> H:=SmallGroup(64,134);;
  gap> N:=NormalSubgroups(H)[15];;
  gap> A:=Centre(GOuterGroup(H,N));;
  gap> G:=ActingGroup(A);;
  gap> R:=ResolutionFiniteGroup(G,3);;
  gap> C:=HomToGModule(R,A);;
  gap> Cohomology(C,2);
  [ 2 ]
```

The following additional commands return a standard 2-cocycle $f : G \times G \to A = C_2 \times C_2$ corresponding to the non-trivial element in $H^2(G,A)$. The value $f(g,h)$ of the 2-cocycle is calculated for all 16 pairs $g, h \in G$.

```
———————————————— Example ————————————————
gap> CH:=CohomologyModule(C,2);;
gap> Elts:=Elements(ActedGroup(CH));
[ <identity> of ..., f1 ]

gap> x:=Elts[2];;
gap> c:=CH!.representativeCocycle(x);
Standard 2-cocycle

gap> f:=Mapping(c);;
gap> for g in G do for h in G do
> Print(f(g,h),"\n");
> od;
> od;
<identity> of ...
<identity> of ...
<identity> of ...
<identity> of ...
<identity> of ...
f6
<identity> of ...
f6
<identity> of ...
<identity> of ...
<identity> of ...
<identity> of ...
<identity> of ...
f6
<identity> of ...
f6
```

The following commands will then construct and identify all extensions of N by G corresponding to the given outer action of G on N.

```
———————————————— Example ————————————————
gap> H := SmallGroup(64,134);;
gap> N := NormalSubgroups(H)[15];;
gap> ON := GOuterGroup(H,N);;
gap> A := Centre(ON);;
gap> G:=ActingGroup(A);;
gap> R:=ResolutionFiniteGroup(G,3);;
gap> C:=HomToGModule(R,A);;
gap> CH:=CohomologyModule(C,2);;
gap> Elts:=Elements(ActedGroup(CH));;

gap> lst := List(Elts{[1..Length(Elts)]},x->CH!.representativeCocycle(x));;
gap> ccgrps := List(lst, x->CcGroup(ON, x));;
gap> #So ccgrps is a list of groups, each being an extension of G by N, corresponding
gap> #to the two elements in H^2(G,A).
```

```
gap> #The following command produces the GAP identification number for each group.
gap> L:=List(ccgrps,IdGroup);
[ [ 64, 134 ], [ 64, 135 ] ]
```

SECOND EXAMPLE

The following example illustrates how to construct a cohomology class k in $H^2(G,A)$ from a cocycle $f : G \times G \to A$, where $G = SL_2(\mathbb{Z}_4)$ and $A = \mathbb{Z}_8$ with trivial action.

──────── Example ────────
```
gap> #We'll construct G=SL(2,Z_4) as a permutation group.
gap> G:=SL(2,ZmodnZ(4));;
gap> G:=Image(IsomorphismPermGroup(G));;

gap> #We'll construct Z_8=Z/8Z as a G-outer group
gap> z_8:=Group((1,2,3,4,5,6,7,8));;
gap> Z_8:=TrivialGModuleAsGOuterGroup(G,z_8);;

gap> #We'll compute the group h=H^2(G,Z_8)
gap> R:=ResolutionFiniteGroup(G,3);;  #R is a free resolution
gap> C:=HomToGModule(R,Z_8);; # C is a chain complex
gap> H:=CohomologyModule(C,2);; #H is the second cohomology H^2(G,Z_8)
gap> h:=ActedGroup(H);; #h is the underlying group of H

gap> #We'll compute  cocycles c2, c5 for the second and fifth cohomology classs
gap> c2:=H!.representativeCocycle(Elements(h)[2]);
Standard 2-cocycle

gap> c5:=H!.representativeCocycle(Elements(h)[5]);
Standard 2-cocycle

gap> #Now we'll construct the cohomology classes C2, C5 in the group h corresponding to the cocyc
gap> C2:=CohomologyClass(H,c2);;
gap> C5:=CohomologyClass(H,c5);;

gap> #Finally, we'll show that C2, C5 are distinct cohomology classes, both of order 4.
gap> C2=C5;
false
gap> Order(C2);
4
gap> Order(C5);
4
```

6.7 Second group cohomology and cocyclic Hadamard matrices

An *Hadamard matrix* is a square $n \times n$ matrix H whose entries are either $+1$ or -1 and whose rows are mutually orthogonal, that is $HH^t = nI_n$ where H^t denotes the transpose and I_n denotes the $n \times n$ identity matrix.

Given a group $G = \{g_1, g_2, \ldots, g_n\}$ of order n and the abelian group $A = \{1, -1\}$ of square roots of unity, any 2-cocycle $f: G \times G \to A$ corresponds to an $n \times n$ matrix $F = (f(g_i, g_j))_{1 \leq i, j \leq n}$ whose entries are ± 1. If F is Hadamard it is called a *cocyclic Hadamard matrix* corresponding to G.

The following commands compute all 192 of the cocyclic Hadamard matrices for the abelian group $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ of order $n = 16$.

```
———————————————————— Example ————————————————————
gap> G:=AbelianGroup([4,4]);;
gap> F:=CocyclicHadamardMatrices(G);;
gap> Length(F);
192
```

6.8 Third group cohomology and homotopy 2-types

HOMOTOPY 2-TYPES

The third cohomology $H^3(G, A)$ of a group G with coefficients in a G-module A, together with the corresponding 3-cocycles, can be used to classify homotopy 2-types. A *homotopy 2-type* is a CW-complex whose homotopy groups are trivial in dimensions $n = 0$ and $n > 2$. There is an equivalence between the two categories

1. (Homotopy category of connected CW-complexes X with trivial homotopy groups $\pi_n(X)$ for $n > 2$)

2. (Localization of the category of simplicial groups with Moore complex of length 1, where localization is with respect to homomorphisms inducing isomorphisms on homotopy groups)

which reduces the homotopy theory of 2-types to a 'computable' algebraic theory. Furthermore, a simplicial group with Moore complex of length 1 can be represented by a group H endowed with two endomorphisms $s: H \to H$ and $t: H \to H$ satisfying the axioms

- $ss = s$, $ts = s$,

- $tt = t$, $st = t$,

- $[\ker s, \ker t] = 1$.

Ths triple (H, s, t) was termed a *cat^1-group* by J.-L. Loday since it can be regarded as a group H endowed with one compatible category structure.

The *homotopy groups* of a cat^1-group H are defined as: $\pi_1(H) = \text{image}(s)/t(\ker(s))$; $\pi_2(H) = \ker(s) \cap \ker(t)$; $\pi_n(H) = 0$ for $n > 2$ or $n = 0$. Note that $\pi_2(H)$ is a $\pi_1(H)$-module where the action is induced by conjugation in H.

A homotopy 2-type X can be represented by a cat^1-group H or by the homotopy groups $\pi_1 X = \pi_1 H$, $\pi_2 X = \pi_2 H$ and a cohomology class $k \in H^3(\pi_1 X, \pi_2 X)$. This class k is the *Postnikov invariant*.

RELATION TO GROUP THEORY

A number of standard group-theoretic constructions can be viewed naturally as a cat^1-group.

1. A $\mathbb{Z}G$-module A can be viewed as a cat^1-group (H, s, t) where H is the semi-direct product $A \rtimes G$ and $s(a, g) = (1, g)$, $t(a, g) = (1, g)$. Here $\pi_1(H) = G$ and $\pi_2(H) = A$.

2. A group G with normal subgroup N can be viewed as a cat^1-group (H, s, t) where H is the semi-direct product $N \rtimes G$ and $s(n, g) = (1, g)$, $t(n, g) = (1, ng)$. Here $\pi_1(H) = G/N$ and $\pi_2(H) = 0$.

3. The homomorphism $\iota \colon G \to Aut(G)$ which sends elements of a group G to the corresponding inner automorphism can be viewed as a cat^1-group (H, s, t) where H is the semi-direct product $G \rtimes Aut(G)$ and $s(g, a) = (1, a)$, $t(g, a) = (1, \iota(g)a)$. Here $\pi_1(H) = Out(G)$ is the outer automorphism group of G and $\pi_2(H) = Z(G)$ is the centre of G.

These three constructions are implemented in HAP.

EXAMPLE

The following commands begin by constructing the cat^1-group H of Construction 3 for the group $G = SmallGroup(64, 134)$. They then construct the fundamental group of H and the second homotopy group of as a π_1-module. These homotopy groups have orders 8 and 2 respectively.

```
                              Example
gap> G:=SmallGroup(64,134);;
gap> H:=AutomorphismGroupAsCatOneGroup(G);;
gap> pi_1:=HomotopyGroup(H,1);;
gap> pi_2:=HomotopyModule(H,2);;
gap> Order(pi_1);
8
gap> Order(ActedGroup(pi_2));
2
```

The following additional commands show that there are 1024 Yoneda equivalence classes of cat^1-groups with fundamental group π_1 and π_1- module equal to π_2 in our example.

```
                              Example
gap> R:=ResolutionFiniteGroup(pi_1,4);;
gap> C:=HomToGModule(R,pi_2);;
gap> CH:=CohomologyModule(C,3);;
gap> AbelianInvariants(ActedGroup(CH));
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

A 3-cocycle $f \colon \pi_1 \times \pi_1 \times \pi_1 \to \pi_2$ corresponding to a random cohomology class $k \in H^3(\pi_1, \pi_2)$ can be produced using the following command.

```
                              Example
gap> x:=Random(Elements(ActedGroup(CH)));;
gap> f:=CH!.representativeCocycle(x);
Standard 3-cocycle
```

The 3-cocycle corresponding to the Postnikov invariant of H itself can be easily constructed directly from its definition in terms of a set-theoretic 'section' of the crossed module corresponding to H.

Chapter 7

Cohomology of groups

7.1 Finite groups

It is possible to compute the low degree (co)homology of a finite group or monoid of small order directly from the bar resolution. The following commands take this approach to computing the fifth integral homology

$$H_5(Q_4, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

of the quaternion group $G = Q_4$ of order 8.

```
──────────────────────────── Example ────────────────────────────
gap> Q:=QuaternionGroup(8);;
gap> B:=BarComplexOfMonoid(Q,6);;
gap> C:=ContractedComplex(B);;
gap> Homology(C,5);
[ 2, 2 ]


gap> List([0..6],B!.dimension);
[ 1, 7, 49, 343, 2401, 16807, 117649 ]
gap> List([0..6],C!.dimension);
[ 1, 2, 2, 1, 2, 4, 102945 ]
```

However, this approach is of limited applicability since the bar resolution involves $|G|^k$ free generators in degree k. A range of techniques, tailored to specific classes of groups, can be used to compute the (co)homology of larger finite groups.

The following example computes the fourth integral cohomomogy of the Mathieu group M_{24}.
$$H^4(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$$

```
──────────────────────────── Example ────────────────────────────
gap> GroupCohomology(MathieuGroup(24),4);
[ 4, 3 ]
```

The following example computes the third integral homology of the Weyl group $W = Weyl(E_8)$, a group of order 696729600.
$$H_3(Weyl(E_8), \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$$

```
                          ──────────── Example ────────────
  gap> L:=SimpleLieAlgebra("E",8,Rationals);;
  gap> W:=WeylGroup(RootSystem(L));;
  gap> Order(W);
  696729600
  gap> GroupHomology(W,3);
  [ 2, 2, 4, 3 ]
```

The preceding calculation could be achieved more quickly by noting that $W = Weyl(E_8)$ is a Coxeter group, and by using the associated Coxeter polytope. The following example uses this approach to compute the fourth integral homology of W. It begins by displaying the Coxeter diagram of W, and then computes

$H_4(Weyl(E_8), \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$

```
                          ──────────── Example ────────────
  gap> D:=[[1,[2,3]],[2,[3,3]],[3,[4,3],[5,3]],[5,[6,3]],[6,[7,3]],[7,[8,3]]];;
  gap> CoxeterDiagramDisplay(D);
```

```
                          ──────────── Example ────────────
  gap> polytope:=CoxeterComplex_alt(D,5);;
  gap> R:=FreeGResolution(polytope,5);
  Resolution of length 5 in characteristic 0 for <matrix group with
  8 generators> .
  No contracting homotopy available.

  gap> C:=TensorWithIntegers(R);
  Chain complex of length 5 in characteristic 0 .

  gap> Homology(C,4);
  [ 2, 2, 2, 2 ]
```

The following example computes the sixth mod-2 homology of the Sylow 2-subgroup $Syl_2(M_{24})$ of the Mathieu group M_{24}.

$H_6(Syl_2(M_{24}), \mathbb{Z}_2) = \mathbb{Z}_2^{143}$

```
                          ──────────── Example ────────────
  gap> GroupHomology(SylowSubgroup(MathieuGroup(24),2),6,2);
  [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
    2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
    2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
    2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
    2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
    2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
    2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

The following example computes the sixth mod-2 homology of the Unitary group $U_3(4)$ of order 312000.

$H_6(U_3(4), \mathbb{Z}_2) = \mathbb{Z}_2^4$

```
_____ Example _____
gap> G:=GU(3,4);;
gap> Order(G);
312000
gap> GroupHomology(G,6,2);
[ 2, 2, 2, 2 ]
```

The following example constructs the Poincare series

$p(x) = \frac{1}{-x^3+3*x^2-3*x+1}$

for the cohomology $H^*(Syl_2(M_{12}, \mathbb{F}_2))$. The coefficient of x^n in the expansion of $p(x)$ is equal to the dimension of the vector space $H^n(Syl_2(M_{12}, \mathbb{F}_2))$. The computation involves SINGULAR's Groebner basis algorithms and the Lyndon-Hochschild-Serre spectral sequence.

```
_____ Example _____
gap> G:=SylowSubgroup(MathieuGroup(12),2);;
gap> P:=PoincareSeriesLHS(G);
(1)/(-x_1^3+3*x_1^2-3*x_1+1)
```

The additional following command uses the Poincare series

```
_____ Example _____
gap> RankHomologyPGroup(G,P,1000);
251000
```

to determine that $H_{1000}(Syl_2(M_{12}, \mathbb{Z}))$ is a direct sum of 251000 non-trivial cyclic 2-groups.

The following example constructs the series

$p(x) = \frac{x^4-x^3+x^2-x+1}{x^6-x^5+x^4-2*x^3+x^2-x+1}$

whose coefficient of x^n is equal to the dimension of the vector space $H^n(M_{11}, \mathbb{F}_2)$ for all n in the range $0 \le n \le 14$. The coefficient is not guaranteed correct for $n \ge 15$.

```
_____ Example _____
gap> PoincareSeriesPrimePart(MathieuGroup(11),2,14);
(x_1^4-x_1^3+x_1^2-x_1+1)/(x_1^6-x_1^5+x_1^4-2*x_1^3+x_1^2-x_1+1)
```

7.2 Nilpotent groups

The following example computes

$H_4(N, \mathbb{Z}) = (Z_3)^4 \oplus \mathbb{Z}^{84}$

for the free nilpotent group N of class 2 on four generators.

```
_____ Example _____
gap> F:=FreeGroup(4);; N:=NilpotentQuotient(F,2);;
gap> GroupHomology(N,4);
[ 3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

7.3 Crystallographic and Almost Crystallographic groups

The following example computes
$$H_5(G,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$
for the 3-dimensional crystallographic space group G with Hermann-Mauguin symbol "P62"

```
——— Example ———
gap> GroupHomology(SpaceGroupBBNWZ("P62"),5);
[ 2, 2 ]
```

The following example computes
$$H^5(G,\mathbb{Z}) = \mathbb{Z}$$
for an almost crystallographic group.

```
——— Example ———
gap> G:=AlmostCrystallographicPcpGroup( 4, 50, [ 1, -4, 1, 2 ] );;
gap> GroupCohomology(G,4);
[ 0 ]
```

7.4 Arithmetic groups

The following example computes
$$H_6(SL_2(\mathcal{O},\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$$
for $\mathcal{O}$ the ring of integers of the number field $\mathbb{Q}(\sqrt{-2})$.

```
——— Example ———
gap> C:=ContractibleGcomplex("SL(2,O-2)");;
gap> R:=FreeGResolution(C,7);;
gap> Homology(TensorWithIntegers(R),6);
[ 2, 12 ]
```

7.5 Artin groups

The following example computes
$$H_5(G,\mathbb{Z}) = \mathbb{Z}_3$$
for G the classical braid group on eight strings.

```
——— Example ———
gap> D:=[[1,[2,3]],[2,[3,3]],[3,[4,3]],[4,[5,3]],[5,[6,3]],[6,[7,3]]];;
gap> CoxeterDiagramDisplay(D);;
```

```
——— Example ———
gap> R:=ResolutionArtinGroup(D,6);;
gap> C:=TensorWithIntegers(R);;
gap> Homology(C,5);
[ 3 ]
```

7.6 Graphs of groups

The following example computes
$$H_5(G,\mathbb{Z}) = \mathbb{Z}_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$$
for G the graph of groups corresponding to the amalgamated product $G = S_5 *_{S_3} S_4$ of the symmetric groups S_5 and S_4 over the canonical subgroup S_3.

```
———————————— Example ————————————
gap> S5:=SymmetricGroup(5);SetName(S5,"S5");
gap> S4:=SymmetricGroup(4);SetName(S4,"S4");
gap> A:=SymmetricGroup(3);SetName(A,"S3");
gap> AS5:=GroupHomomorphismByFunction(A,S5,x->x);
gap> AS4:=GroupHomomorphismByFunction(A,S4,x->x);
gap> D:=[S5,S4,[AS5,AS4]];
gap> GraphOfGroupsDisplay(D);
```

```
———————————— Example ————————————
gap> R:=ResolutionGraphOfGroups(D,6);;
gap> Homology(TensorWithIntegers(R),5);
[ 2, 2, 2, 2, 2 ]
```

7.7 Cohomology with coefficients in a module

There are various ways to represent a $\mathbb{Z}G$-module A with action $G \times A \to A, (g,a) \mapsto \alpha(g,a)$.

One possibility is to use the data type of a *G-Outer Group* which involves three components: an *ActedGroup* A; an *ActingGroup* G; a *Mapping* $(g,a) \mapsto \alpha(g,a)$. The following example uses this data type to compute the cohomology $H^4(G,A) = \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$ of the symmetric group $G = S_6$ with coefficients in the integers $A = \mathbb{Z}$ where odd permutations act non-trivially on A.

```
———————————— Example ————————————
gap> G:=SymmetricGroup(6);;

gap> A:=AbelianPcpGroup([0]);;
gap> alpha:=function(g,a); return a^SignPerm(g); end;;
gap> A:=GModuleAsGOuterGroup(G,A,alpha);
ZG-module with abelian invariants [ 0 ] and G= SymmetricGroup( [ 1 .. 6 ] )

gap> R:=ResolutionFiniteGroup(G,5);;
gap> C:=HomToGModule(R,A);
G-cocomplex of length 5 .

gap> Cohomology(C,4);
[ 2, 2, 5 ]
```

If $A = \mathbb{Z}^n$ and G acts as
$$G \times A \to A, (g,v) \mapsto \rho(g)\,v$$
where $\rho: G \to Gl_n(\mathbb{Z})$ is a (not necessarily faithful) matrix representation of degree n then we can avoid the use of G-outer groups and use just the homomorphism ρ instead. The following example uses this data type to compute the cohomology

$H^6(G,A) = \mathbb{Z}_2$

and the homology

$H_6(G,A) = 0$

of the alternating group $G = A_5$ with coefficients in $A = \mathbb{Z}^5$ where elements of G act on $\mathbb{Z}^5$ via an irreducible representation.

``` Example
gap> G:=AlternatingGroup(5);;
gap> rho:=IrreducibleRepresentations(G)[5];
[ (1,2,3,4,5), (3,4,5) ] ->
[
  [ [ 0, 0, 1, 0, 0 ], [ -1, -1, 0, 0, 1 ], [ 0, 1, 1, 1, 0 ],
      [ 1, 0, -1, 0, -1 ], [ -1, -1, 0, -1, 0 ] ],
  [ [ -1, -1, 0, 0, 1 ], [ 1, 0, -1, 0, -1 ], [ 0, 0, 0, 0, 1 ],
      [ 0, 0, 1, 0, 0 ], [ 0, 0, 0, 1, 0 ] ] ]
gap> R:=ResolutionFiniteGroup(G,7);;
gap> C:=HomToIntegralModule(R,rho);;
gap> Cohomology(C,6);
[ 2 ]
gap> D:=TensorWithIntegralModule(R,rho);
Chain complex of length 7 in characteristic 0 .

gap> Homology(D,6);
[  ]
```

If $V = K^d$ is a vetor space of dimension d over the field $K = GF(p)$ with p a prime and G acts on V via a homomorphism $\rho \colon G \to GL_d(K)$ then the homology $H^n(G,V)$ can again be computed without the use of G-outer groups. As an example, the following commands compute

$H^4(GL(3,2),V) = K^2$

where $K = GF(2)$ and $GL(3,2)$ acts with its natural action on $V = K^3$.

``` Example
gap> G:=GL(3,2);;
gap> rho:=GroupHomomorphismByFunction(G,G,x->x);;
gap> R:=ResolutionFiniteGroup(G,5);;
gap> C:=HomToModPModule(R,rho);;
gap> Cohomology(C,4);
2
```

It can be computationally difficult to compute high degree terms in resolutions for large finite groups. But the p-primary part of the homology can be computed using resolutions of Sylow p-subgroups. This approach is used in the following example that computes the 2-primary part

$H_{11}(A_7,A)_{(2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

of the degree 11 homology of the alternating group A_7 of degree 7 with coefficients in the module $A = \mathbb{Z}^7$ on which A_7 acts by permuting basis vectors.

``` Example
gap> G:=AlternatingGroup(7);;
gap> rho:=PermToMatrixGroup(G);;
gap> R:=ResolutionFiniteGroup(SylowSubgroup(G,2),12);;
gap> F:=function(X); return TensorWithIntegralModule(X,rho); end;;
```

```
gap> PrimePartDerivedFunctor(G,R,F,11);
[ 2, 2, 4 ]
```

Similar commands compute

$H_3(A_{10},A)_{(2)} = \mathbb{Z}_4$

with coefficient module $A = \mathbb{Z}^{10}$ on which A_{10} acts by permuting basis vectors.

```
────────────────── Example ──────────────────
gap> G:=AlternatingGroup(10);;
gap> rho:=PermToMatrixGroup(G);;
gap> R:=ResolutionFiniteGroup(SylowSubgroup(G,2),4);;
gap> F:=function(X); return TensorWithIntegralModule(X,rho); end;;
gap> PrimePartDerivedFunctor(G,R,F,3);
[ 4 ]
```

The following commands compute

$H_{100}(GL(3,2),V) = K^{34}$

where V is the vector space of dimension 3 over $K = GF(2)$ acting via some irreducible representation $\rho\colon GL(3,2) \to GL(V)$.

```
────────────────── Example ──────────────────
gap> G:=GL(3,2);;
gap> rho:=IrreducibleRepresentations(G,GF(2))[3];
CompositionMapping( [ (5,7)(6,8), (2,3,5)(4,7,6) ] ->
[ <an immutable 3x3 matrix over GF2>, <an immutable 3x3 matrix over GF2> ],
 <action isomorphism> )
gap> F:=function(X); return TensorWithModPModule(X,rho); end;;
gap> S:=ResolutionPrimePowerGroup(SylowSubgroup(G,2),101);;
gap> PrimePartDerivedFunctor(G,S,F,100);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

7.8 Cohomology as a functor of the first variable

Suppose given a group homomorphism $f\colon G_1 \to G_2$ and a G_2-module A. Then A is naturally a G_1-module with action via f, and there is an induced cohomology homomorphism $H^n(f,A)\colon H^n(G_2,A) \to H^n(G_1,A)$.

The following example computes this cohomology homomorphism in degree $n = 6$ for the inclusion $f\colon A_5 \to S_5$ and $A = \mathbb{Z}^5$ with action that permutes the canonical basis. The final commands determine that the kernel of the homomorphism $H^6(f,A)$ is the Klein group of order 4 and that the cokernel is cyclic of order 6.

```
────────────────── Example ──────────────────
gap> G1:=AlternatingGroup(5);;
gap> G2:=SymmetricGroup(5);;
gap> f:=GroupHomomorphismByFunction(G1,G2,x->x);;
gap> pi:=PermToMatrixGroup(G2,5);;
gap> R1:=ResolutionFiniteGroup(G1,7);;
gap> R2:=ResolutionFiniteGroup(G2,7);;
```

```
gap> F:=EquivariantChainMap(R1,R2,f);;
gap> C:=HomToIntegralModule(F,pi);;
gap> c:=Cohomology(C,6);
[ g1, g2, g3 ] -> [ id, id, g3 ]

gap> AbelianInvariants(Kernel(c));
[ 2, 2 ]
gap> AbelianInvariants(Range(c)/Image(c));
[ 2, 3 ]
```

7.9 Cohomology as a functor of the second variable and the long exact coefficient sequence

A short exact sequence of $\mathbb{Z}G$-modules $A \rightarrowtail B \twoheadrightarrow C$ induces a long exact sequence of cohomology groups

$$\to H^n(G,A) \to H^n(G,B) \to H^n(G,C) \to H^{n+1}(G,A) \to .$$

Consider the symmetric group $G = S_4$ and the sequence $\mathbb{Z}_4 \rightarrowtail \mathbb{Z}_8 \twoheadrightarrow \mathbb{Z}_2$ of trivial $\mathbb{Z}G$-modules. The following commands compute the induced cohomology homomorphism

$$f : H^3(S_4, \mathbb{Z}_4) \to H^3(S_4, \mathbb{Z}_8)$$

and determine that the image of this induced homomorphism has order 8 and that its kernel has order 2.

```
—————————————————— Example ——————————————————
gap> G:=SymmetricGroup(4);;
gap> x:=(1,2,3,4,5,6,7,8);;
gap> a:=Group(x^2);;
gap> b:=Group(x);;
gap> ahomb:=GroupHomomorphismByFunction(a,b,y->y);;
gap> A:=TrivialGModuleAsGOuterGroup(G,a);;
gap> B:=TrivialGModuleAsGOuterGroup(G,b);;
gap> phi:=GOuterGroupHomomorphism();;
gap> phi!.Source:=A;;
gap> phi!.Target:=B;;
gap> phi!.Mapping:=ahomb;;

gap> Hphi:=CohomologyHomomorphism(phi,3);;

gap> Size(ImageOfGOuterGroupHomomorphism(Hphi));
8

gap> Size(KernelOfGOuterGroupHomomorphism(Hphi));
2
```

The following commands then compute the homomorphism

$$H^3(S_4, \mathbb{Z}_8) \to H^3(S_4, \mathbb{Z}_2)$$

induced by $\mathbb{Z}_4 \rightarrowtail \mathbb{Z}_8 \twoheadrightarrow \mathbb{Z}_2$, and determine that the kernel of this homomorphsim has order 8.

```
—————————————————— Example ——————————————————
gap> bhomc:=NaturalHomomorphismByNormalSubgroup(b,a);
gap> B:=TrivialGModuleAsGOuterGroup(G,b);
```

```
gap> C:=TrivialGModuleAsGOuterGroup(G,Image(bhomc));
gap> psi:=GOuterGroupHomomorphism();
gap> psi!.Source:=B;
gap> psi!.Target:=C;
gap> psi!.Mapping:=bhomc;

gap> Hpsi:=CohomologyHomomorphism(psi,3);

gap> Size(KernelOfGOuterGroupHomomorphism(Hpsi));
8
```

The following commands then compute the connecting homomorphism
$H^2(S_4, \mathbb{Z}_2) \to H^3(S_4, \mathbb{Z}_4)$
and determine that the image of this homomorphism has order 2.

```
—————————————— Example ——————————————
gap> delta:=ConnectingCohomologyHomomorphism(psi,2);;
gap> Size(ImageOfGOuterGroupHomomorphism(delta));
```

Note that the various orders are consistent with exactness of the sequence
$H^2(S_4, \mathbb{Z}_2) \to H^3(S_4, \mathbb{Z}_4) \to H^3(S_4, \mathbb{Z}_8) \to H^3(S_4, \mathbb{Z}_2)$.

7.10 Transfer Homomorphism

Consider the action of the symmetric group $G = S_5$ on $A = \mathbb{Z}^5$ which permutes the canonical basis. The action restricts to the sylow 2-subgroup $P = Syl_2(G)$. The following commands compute the cohomology transfer homomorphism $t^4: H^4(P, A) \to H^4(S_5, A)$ and determine its kernel and image. The integral homology transfer $t_4: H_4(S_5, \mathbb{Z}) \to H_5(P, \mathbb{Z})$ is also computed.

```
—————————————— Example ——————————————
gap> G:=SymmetricGroup(5);;
gap> P:=SylowSubgroup(G,2);;
gap> R:=ResolutionFiniteGroup(G,5);;
gap> A:=PermToMatrixGroup(G);;
gap> tr:=TransferCochainMap(R,P,A);
Cochain Map between complexes of length 5 .

gap> t4:=Cohomology(tr,4);
[ g1, g2, g3, g4 ] -> [ id, g1, g2, g4 ]
gap> StructureDescription(Kernel(t4));
"C2 x C2"
gap> StructureDescription(Image(t4));
"C4 x C2"

gap> tr:=TransferChainMap(R,P);
Chain Map between complexes of length 5 .

gap> Homology(tr,4);
[ g1 ] -> [ g1 ]
```

7.11 Cohomology rings of finite fundamental groups of 3-manifolds

A *spherical 3-manifold* is a 3-manifold arising as the quotient S^3/Γ of the 3-sphere S^3 by a finite subgroup Γ of $SO(4)$ acting freely as rotations. The geometrization conjecture, proved by Grigori Perelman, implies that every closed connected 3-manifold with a finite fundamental group is homeomorphic to a spherical 3-manifold.

A spherical 3-manifold S^3/Γ has finite fundamental group isomorphic to Γ. This fundamental group is one of:

- $\Gamma = C_m = \langle x \mid x^m \rangle$ (CYCLIC FUNDAMENTAL GROUP)

- $\Gamma = C_m \times \langle x, y \mid xyx^{-1} = y^{-1}, x^{2^k} = y^n \rangle$ for integers $k, m \geq 1, n \geq 2$ and m coprime to $2n$ (PRISM MANIFOLD CASE)

- $\Gamma = C_m \times \langle x, y, z \mid (xy)^2 = x^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 \rangle$ for integers $k, m \geq 1$ and m coprime to 6 (TETRAHEDRAL CASE)

- $\Gamma = C_m \times \langle x, y \mid (xy)^2 = x^3 = y^4 \rangle$ for $m \geq 1$ coprime to 6 (OCTAHEDRAL CASE)

- $\Gamma = C_m \times \langle x, y \mid (xy)^2 = x^3 = y^5 \rangle$ for $m \geq 1$ coprime to 30 (ICOSAHEDRAL CASE).

This list of cases is taken from the Wikipedia pages. The group Γ has periodic cohomology since it acts on a sphere. The cyclic group has period 2 and in the other four cases it has period 4. (Recall that in general a finite group G has *periodic cohomology of period n* if there is an element $u \in H^n(G, \mathbb{Z})$ such that the cup product $- \cup u : H^k(G, \mathbb{Z}) \to H^{k+n}(G, \mathbb{Z})$ is an isomorphism for all $k \geq 1$. It can be shown that G has periodic cohomology of period n if and only if $H^n(G, \mathbb{Z}) = \mathbb{Z}_{|G|}$.)

The cohomology of the cyclic group is well-known, and the cohomology of a direct product can be obtained from that of the factors using the Kunneth formula.

In the icosahedral case with $m = 1$ the following commands yield $$H^\ast(\Gamma, \mathbb{Z}) = Z[t]/(120t = 0)$$ with generator t of degree 4. The final command demonstrates that a periodic resolution is used in the computation.

```
                                    Example
  gap> F:=FreeGroup(2);;x:=F.1;;y:=F.2;;
  gap> G:=F/[(x*y)^2*x^-3, x^3*y^-5];;
  gap> Order(G);
  120
  gap> R:=ResolutionSmallGroup(G,5);;
  gap> n:=0;;Cohomology(HomToIntegers(R),n);
  [ 0 ]
  gap> n:=1;;Cohomology(HomToIntegers(R),n);
  [  ]
  gap> n:=2;;Cohomology(HomToIntegers(R),n);
  [  ]
  gap> n:=3;;Cohomology(HomToIntegers(R),n);
  [  ]
  gap> n:=4;;Cohomology(HomToIntegers(R),n);
  [ 120 ]

  gap> List([0..5],k->R!.dimension(k));
  [ 1, 2, 2, 1, 1, 2 ]
```

In the octahedral case with $m = 1$ we obtain $$H^\ast(\Gamma,\mathbb Z) = \mathbb Z[s,t]/(s^2=24t, 2s=0, 48t=0)$$ where s has degree 2 and t has degree 4, from the following commands.

```
──────── Example ────────
gap> F:=FreeGroup(2);;x:=F.1;;y:=F.2;;
gap> G:=F/[(x*y)^2*x^-3, x^3*y^-4];;
gap> Order(G);
48
gap> R:=ResolutionFiniteGroup(G,5);;
gap> n:=0;;Cohomology(HomToIntegers(R),n);
[ 0 ]
gap> n:=1;;Cohomology(HomToIntegers(R),n);
[ ]
gap> n:=2;;Cohomology(HomToIntegers(R),n);
[ 2 ]
gap> n:=3;;Cohomology(HomToIntegers(R),n);
[ ]
gap> n:=4;;Cohomology(HomToIntegers(R),n);
[ 48 ]
gap> IntegralCupProduct(R,[1],[1],2,2);
[ 24 ]
```

In the tetrahedral case with $m = 1$ we obtain $$H^\ast(\Gamma,\mathbb Z) = \mathbb Z[s,t]/(s^2=16t, 3s=0, 24t=0)$$ where s has degree 2 and t has degree 4, from the following commands.

```
──────── Example ────────
gap> F:=FreeGroup(3);;x:=F.1;;y:=F.2;;z:=F.3;;
gap> G:=F/[(x*y)^2*x^-2, x^2*y^-2, z*x*z^-1*y^-1, z*y*z^-1*y^-1*x^-1,z^3];;
gap> Order(G);
24
gap> R:=ResolutionFiniteGroup(G,5);;
gap> n:=1;;Cohomology(HomToIntegers(R),n);
[ ]
gap> n:=2;;Cohomology(HomToIntegers(R),n);
[ 3 ]
gap> n:=3;;Cohomology(HomToIntegers(R),n);
[ ]
gap> n:=4;;Cohomology(HomToIntegers(R),n);
[ 24 ]
gap> IntegralCupProduct(R,[1],[1],2,2);
[ 16 ]
```

A theoretical calculation of the integral and mod-p cohomology rings of all of these fundamental groups of spherical 3-manifolds is given in [TZ08].

7.12 Explicit cocycles

Given a $\mathbb Z G$-resolution R_* and a $\mathbb Z G$-module A, one defines an *n-cocycle* to be a $\mathbb Z G$-homomorphism $f: R_n \to A$ for which the composite homomorphism $f d_{n+1}: R_{n+1} \to A$ is zero. If R_* happens to be

the standard bar resolution (i.e. the cellular chain complex of the nerve of the group G considered as a one object category) then the free $\mathbb{Z}G$-generators of R_n are indexed by n-tuples $(g_1|g_2|\ldots|g_n)$ of elements g_i in G. In this case we say that the n-cocycle is a *standard n-cocycle* and we think of it as a set-theoretic function

$f: G \times G \times \cdots \times G \longrightarrow A$

satisfying a certain algebraic cocycle condition. Bearing in mind that a standard n-cocycle really just assigns an element $f(g_1,\ldots,g_n) \in A$ to an n-simplex in the nerve of G, the cocycle condition is a very natural one which states that *f must vanish on the boundary of a certain $(n+1)$-simplex*. For $n = 2$ the condition is that a 2-cocycle $f(g_1,g_2)$ must satisfy

$g.f(h,k) + f(g,hk) = f(gh,k) + f(g,h)$

for all $g,h,k \in G$. This equation is explained by the following picture.

The definition of a cocycle clearly depends on the choice of $\mathbb{Z}G$-resolution R_*. However, the cohomology group $H^n(G,A)$, which is a group of equivalence classes of n-cocycles, is independent of the choice of R_*.

There are some occasions when one needs explicit examples of standard cocycles. For instance:

- Let G be a finite group and k a field of characteristic 0. The group algebra $k(G)$, and the algebra $F(G)$ of functions $d_g: G \to k, h \to d_{g,h}$, are both Hopf algebras. The tensor product $F(G) \otimes k(G)$ also admits a Hopf algebra structure known as the quantum double $D(G)$. A twisted quantum double $D_f(G)$ was introduced by R. Dijkraaf, V. Pasquier & P. Roche [DPR91]. The twisted double is a quasi-Hopf algebra depending on a 3-cocycle $f: G \times G \times G \to k$. The multiplication is given by $(d_g \otimes x)(d_h \otimes y) = d_{gx,xh}\beta_g(x,y)(d_g \otimes xy)$ where β_a is defined by $\beta_a(h,g) = f(a,h,g)f(h,h^{-1}ah,g)^{-1}f(h,g,(hg)^{-1}ahg)$. Although the algebraic structure of $D_f(G)$ depends very much on the particular 3-cocycle f, representation-theoretic properties of $D_f(G)$ depend only on the cohomology class of f.

- An explicit 2-cocycle $f: G \times G \to A$ is needed to construct the multiplication $(a,g)(a',g') = (a + g \cdot a' + f(g,g'), gg')$ in the extension a group G by a $\mathbb{Z}G$-module A determined by the cohomology class of f in $H^2(G,A)$. See 6.6.

- In work on coding theory and Hadamard matrices a number of papers have investigated square matrices (a_{ij}) whose entries $a_{ij} = f(g_i,g_j)$ are the values of a 2-cocycle $f: G \times G \to \mathbb{Z}_2$ where G is a finite group acting trivially on $\mathbb{Z}_2$. See for instance [Hor00] and 6.7.

Given a $\mathbb{Z}G$-resolution R_* (with contracting homotopy) and a $\mathbb{Z}G$-module A one can use HAP commands to compute explicit standard n-cocycles $f: G^n \to A$. With the twisted quantum double in mind, we illustrate the computation for $n = 3$, $G = S_3$, and $A = U(1)$ the group of complex numbers of modulus 1 with trivial G-action.

We first compute a $\mathbb{Z}G$-resolution R_*. The Universal Coefficient Theorem gives an isomorphism $H^3(G,U(1)) = Hom_{\mathbb{Z}}(H_3(G,\mathbb{Z}),U(1))$, The multiplicative group $U(1)$ can thus be viewed as $\mathbb{Z}_m$ where m is a multiple of the exponent of $H_3(G,\mathbb{Z})$.

```
─────────────────────────────── Example ───────────────────────────────
  gap> G:=SymmetricGroup(3);;
  gap> R:=ResolutionFiniteGroup(G,4);;
  gap> TR:=TensorWithIntegers(R);;
  gap> Homology(TR,3);
  [ 6 ]
  gap> R!.dimension(3);
  4
```

```
gap> R!.dimension(4);
5
```

We thus replace the very infinite group U(1) by the finite cyclic group $\mathbb{Z}_6$. Since the resolution R_* has 4 generators in degree 3, a homomorphism $f: R^3 \to U(1)$ can be represented by a list $f = [f_1, f_2, f_3, f_4]$ with f_i the image in $\mathbb{Z}_6$ of the ith generator. The cocycle condition on f can be expressed as a matrix equation

$Mf^t = 0 \bmod 6$.

where the matrix M is obtained from the following command and f^t denotes the transpose.

———— Example ————
```
gap> M:=CocycleCondition(R,3);;
```

A particular cocycle $f = [f_1, f_2, f_3, f_4]$ can be obtained by choosing a solution to the equation Mf^t=0.

———— Example ————
```
gap> SolutionsMod2:=NullspaceModQ(TransposedMat(M),2);
[ [ 0, 0, 0, 0 ], [ 0, 0, 1, 1 ], [ 1, 1, 0, 0 ], [ 1, 1, 1, 1 ] ]

gap> SolutionsMod3:=NullspaceModQ(TransposedMat(M),3);
[ [ 0, 0, 0, 0 ], [ 0, 0, 0, 1 ], [ 0, 0, 0, 2 ], [ 0, 0, 1, 0 ],
  [ 0, 0, 1, 1 ], [ 0, 0, 1, 2 ], [ 0, 0, 2, 0 ], [ 0, 0, 2, 1 ],
  [ 0, 0, 2, 2 ] ]
```

A non-standard 3-cocycle f can be converted to a standard one using the command `StandardCocycle(R,f,n,q)`. This command inputs R_*, integers n and q, and an n-cocycle f for the resolution R_*. It returns a standard cocycle $G^n \to \mathbb{Z}_q$.

———— Example ————
```
gap> f:=3*SolutionsMod2[3] - SolutionsMod3[5];   #An example solution to Mf=0 mod 6.
[ 3, 3, -1, -1 ]

gap> Standard_f:=StandardCocycle(R,f,3,6);;

gap> g:=Random(G); h:=Random(G); k:=Random(G);
(1,2)
(1,3,2)
(1,3)

gap> Standard_f(g,h,k);
3
```

A function $f: G \times G \times G \to A$ is a standard 3-cocycle if and only if

$g \cdot f(h,k,l) - f(gh,k,l) + f(g,hk,l) - f(g,h,kl) + f(g,h,k) = 0$

for all $g,h,k,l \in G$. In the above example the group $G = S_3$ acts trivially on $A = Z_6$. The following commands show that the standard 3-cocycle produced in the example really does satisfy this 3-cocycle condition.

```
────────────── Example ──────────────
gap> sf:=Standard_f;;

gap> Test:=function(g,h,k,l);
> return sf(h,k,l) - sf(g*h,k,l) + sf(g,h*k,l) - sf(g,h,k*l) + sf(g,h,k);
> end;
function( g, h, k, l ) ... end

gap> for g in G do for h in G do for k in G do for l in G do
> Print(Test(g,h,k,l),",");
> od;od;od;od;
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,0,6,6,0,0,6,
0,0,0,0,0,6,6,6,0,6,0,12,12,6,12,6,0,12,6,0,6,6,0,0,0,0,0,0,12,12,6,6,6,0,
6,6,0,6,6,0,0,-6,0,0,0,0,0,0,0,0,0,6,6,6,6,6,0,0,0,0,0,0,6,0,0,6,6,0,6,6,
0,6,0,0,6,6,6,0,0,0,0,0,0,-6,0,0,-6,0,-6,0,0,0,0,0,0,0,0,6,6,0,6,0,0,6,0,0,
0,0,0,6,6,6,0,0,0,6,6,6,0,0,0,0,-6,0,6,6,0,0,0,0,0,0,0,12,6,6,0,6,0,0,0,0,12,
6,0,0,0,0,0,0,0,6,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,0,0,6,0,0,6,0,0,0,0,0,6,6,
6,0,0,0,6,12,6,6,0,0,0,-6,0,0,6,0,0,0,0,0,0,0,12,12,6,6,6,0,0,0,0,6,6,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,0,0,6,0,6,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,6,6,6,6,0,
6,6,0,6,6,0,12,12,6,12,12,0,0,0,0,0,0,6,6,0,0,0,0,6,6,6,12,12,0,-6,-6,0,0,
0,0,6,6,0,0,6,0,0,6,0,6,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,0,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,6,
0,6,0,0,6,0,0,0,0,0,0,0,0,0,0,6,6,0,6,0,0,6,0,0,0,0,0,0,0,0,0,0,0,0,0,6,0,
0,0,0,0,0,0,0,0,0,0,0,0,6,0,0,6,6,0,6,6,0,6,0,0,6,6,6,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,6,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,6,0,0,0,0,0,0,6,6,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,-6,0,6,0,6,0,6,0,0,0,0,0,0,0,12,12,6,12,12,0,6,6,0,6,6,0,
0,0,0,0,0,0,12,12,6,12,12,0,6,6,0,6,6,0,0,0,0,0,0,0,0,0,0,0,6,6,6,6,6,0,
0,0,0,0,0,0,0,6,0,0,6,6,0,6,6,0,6,0,0,6,6,6,0,0,0,-6,0,0,0,-6,0,0,-6,0,-6,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,6,6,6,6,0,6,6,0,0,0,0,0,0,0,6,6,0,0,0,
0,0,0,0,6,6,0,-6,0,0,-6,0,0,12,6,0,-6,-6,0,0,0,0,6,6,0,0,6,0,0,6,0,6,6,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,-6,0,0,0,0,0,0,0,0,0,6,6,6,6,6,0,6,12,0,6,0,0,6,0,0,6,0,0,0,0,0,0,0,
0,6,12,0,0,0,0,0,0,6,6,0,-6,-6,0,0,0,0,0,0,0,0,6,0,0,6,0,6,6,0,0,0,0,0,0,0,
6,0,0,0,6,0,0,6,0,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6,
0,0,0,0,0,0,0,0,6,0,0,0,0,0,0,6,6,0,6,6,0,6,6,6,12,12,0,0,0,0,0,0,6,6,0,
6,6,0,6,6,6,12,12,0,0,0,0,0,0,6,6,0,0,6,0,0,6,0,6,6,
```

7.13 Quillen's complex and the p-part of homology

Let G be a finite group with order divisible by prime p. Let $\mathscr{A} = \mathscr{A}_p(G)$ denote Quillen's simplicial complex arising as the order complex of the poset of non-trivial elementary abelian p-subgroups of G. The group G acts on $\mathscr{A}$. Denote the orbit of a k-simplex e^k by $[e^k]$, and the stabilizer of e^k by $Stab(e^k) \leq G$. For a finite abelian group H let H_p denote the Sylow p-subgroup or the "p-part". Peter Webb proved the following.

THEOREM.[Peter Webb] For any G-module M there is a (non natural) homomorphism

$H_n(G,M)_p \oplus \bigoplus_{[e^k]:k \text{ odd}} H_n(Stab(e^k),M)_p \cong \bigoplus_{[e^k]:k \text{ even}} H_n(Stab(e^k),M)_p$

for $n \geq 0$. The isomorphism can also be expressed as

$H_n(G,M)_p \cong \bigoplus_{[e^k]:k \text{ even}} H_n(Stab(e^k),M)_p - \bigoplus_{[e^k]:k \text{ odd}} H_n(Stab(e^k),M)_p$ and terms can often be cancelled.

Thus the additive structure of the p-part of the homology of G is determined by that of the stabilizer groups. The result also holds with homology replaced by cohomology.

ILLUSTRATION 1

As an illustration of the theorem, the following commands calculate

$H_n(M_{12},M)_3 \cong \bigoplus_{1 \leq i \leq 3} H_n(S_i,M)_3 - \bigoplus_{4 \leq i \leq 5} H_n(S_4,M)_3$

for the Mathieu simple group M_{12} of order 95040, where

$S_1 \cong S_3 = (((C_3 \times C_3) : Q_8) : C_3) : C_2$
$S_2 = A_4 \times S_3$
$S_4 = C_3 \times S_3$
$S_5 = ((C_3 \times C_3) : C_3) : (C_2 \times C_2)$.

```
————————————————————————————— Example —————————————————————————————
gap> G:=MathieuGroup(12);;
gap> D:=HomologicalGroupDecomposition(G,3);;
gap> List(D[1],StructureDescription);
[ "(((C3 x C3) : Q8) : C3) : C2", "A4 x S3", "(((C3 x C3) : Q8) : C3) : C2" ]
gap> List(D[2],StructureDescription);
[ "C3 x S3", "((C3 x C3) : C3) : (C2 x C2)" ]
```

ILLUSTRATION 2

As a second illustration, the following commands show that $H_n(M_{23},M)_p$ is periodic for primes $p = 5,7,11,23$ of periods dividing $8,6,10,22$ respectively.

```
————————————————————————————— Example —————————————————————————————
gap> G:=MathieuGroup(23);;
gap> Factors(Order(G));
[ 2, 2, 2, 2, 2, 2, 2, 3, 3, 5, 7, 11, 23 ]

gap> sd:=StructureDescription;;

gap> D:=HomologicalGroupDecomposition(G,5);;
gap> List(D[1],sd);List(D[2],sd);
[ "C15 : C4" ]
[ ]
gap> IsPeriodic(D[1][1]);
true
gap> CohomologicalPeriod(D[1][1]);
8
```

```
gap> D:=HomologicalGroupDecomposition(G,7);;
gap> List(D[1],sd);List(D[2],sd);
[ "C2 x (C7 : C3)" ]
[  ]
gap> IsPeriodic(D[1][1]);
true
gap> CohomologicalPeriod(D[1][1]);
6

gap> D:=HomologicalGroupDecomposition(G,11);;
gap> List(D[1],sd);List(D[2],sd);
[ "C11 : C5" ]
[  ]
gap> IsPeriodic(D[1][1]);
true
gap> CohomologicalPeriod(D[1][1]);
10

gap> D:=HomologicalGroupDecomposition(G,23);;
gap> List(D[1],sd);List(D[2],sd);
[ "C23 : C11" ]
[  ]
gap> IsPeriodic(D[1][1]);
true
gap> CohomologicalPeriod(D[1][1]);
22
```

The order $|M_{23}| = 10200960$ is divisible by primes $p = 2, 3, 5, 7, 11, 23$. For $p = 3$ the following commands establish that the Poincare series

$(x^{16} - 2x^{15} + 3x^{14} - 4x^{13} + 4x^{12} - 4x^{11} + 4x^{10} - 3x^9 + 3x^8 - 3x^7 + 4x^6 - 4x^5 + 4x^4 - 4x^3 + 3x^2 - 2x + 1) / (x^{18} - 2x^{17} + 3x^{16} - 4x^{15} + 4x^{14} - 4x^{13} + 4x^{12} - 4x^{11} + 4x^{10} - 4x^9 + 4x^8 - 4x^7 + 4x^6 - 4x^5 + 4x^4 - 4x^3 + 3x^2 - 2x + 1)$

describes the dimension of the vector space $H^n(M_{23}, \mathbb{Z}_3)$ up to at least degree $n = 40$. To prove that it describes the dimension in all degrees one would need to verify "completion criteria".

```
──────── Example ────────
gap> G:=MathieuGroup(23);;
gap> D:=HomologicalGroupDecomposition(G,3);;
gap> List(D[1],StructureDescription);
[ "(C3 x C3) : QD16", "A5 : S3" ]
gap> List(D[2],StructureDescription);
[ "S3 x S3" ]

gap> P1:=PoincareSeriesPrimePart(D[1][1],3,40);
(x_1^16-2*x_1^15+3*x_1^14-4*x_1^13+4*x_1^12-4*x_1^11+4*x_1^10-3*x_1^9+3*x_1^8-3*x_1^7+4*x_1^6-4*x
4*x_1^4-4*x_1^3+3*x_1^2-2*x_1+1)/(x_1^18-2*x_1^17+3*x_1^16-4*x_1^15+4*x_1^14-4*x_1^13+4*x_1^12-4*
1+4*x_1^10-4*x_1^9+4*x_1^8-4*x_1^7+4*x_1^6-4*x_1^5+4*x_1^4-4*x_1^3+3*x_1^2-2*x_1+1)

gap> P2:=PoincareSeriesPrimePart(D[1][2],3,40);
(x_1^4-2*x_1^3+3*x_1^2-2*x_1+1)/(x_1^6-2*x_1^5+3*x_1^4-4*x_1^3+3*x_1^2-2*x_1+1)

gap> P3:=PoincareSeriesPrimePart(D[2][1],3,40);
```

```
(x_1^4-2*x_1^3+3*x_1^2-2*x_1+1)/(x_1^6-2*x_1^5+3*x_1^4-4*x_1^3+3*x_1^2-2*x_1+1)
```

Chapter 8

Cohomology operations

8.1 Steenrod operations on the classifying space of a finite 2-group

The following example determines a presentation for the cohomology ring $H^*(Syl_2(M_{12}), \mathbb{Z}_2)$. The Lyndon-Hochschild-Serre spectral sequence, and Groebner basis routines from SINGULAR, are used to determine how much of a resolution to compute for the presentation.

```
─────────────────────────────── Example ───────────────────────────────
 gap> G:=SylowSubgroup(MathieuGroup(12),2);;
 gap> Mod2CohomologyRingPresentation(G);
 Graded algebra GF(2)[ x_1, x_2, x_3, x_4, x_5, x_6, x_7 ] /
 [ x_2*x_3, x_1*x_2, x_2*x_4, x_3^3+x_3*x_5,
   x_1^2*x_4+x_1*x_3*x_4+x_3^2*x_4+x_3^2*x_5+x_1*x_6+x_4^2+x_4*x_5,
   x_1^2*x_3^2+x_1*x_3*x_5+x_3^2*x_5+x_3*x_6,
   x_1^3*x_3+x_3^2*x_4+x_3^2*x_5+x_1*x_6+x_3*x_6+x_4*x_5,
   x_1*x_3^2*x_4+x_1*x_3*x_6+x_1*x_4*x_5+x_3*x_4^2+x_3*x_4*x_5+x_3*x_5^\
 2+x_4*x_6, x_1^2*x_3*x_5+x_1*x_3^2*x_5+x_3^2*x_6+x_3*x_5^2,
   x_3^2*x_4^2+x_3^2*x_5^2+x_1*x_5*x_6+x_3*x_4*x_6+x_4*x_5^2,
   x_1*x_3*x_4^2+x_1*x_3*x_4*x_5+x_1*x_3*x_5^2+x_3^2*x_5^2+x_1*x_4*x_6+\
 x_2^2*x_7+x_2*x_5*x_6+x_3*x_4*x_6+x_3*x_5*x_6+x_4^2*x_5+x_4*x_5^2+x_6^\
 2, x_1*x_3^2*x_6+x_3^2*x_4*x_5+x_1*x_5*x_6+x_4*x_5^2,
   x_1^2*x_3*x_6+x_1*x_5*x_6+x_2^2*x_7+x_2*x_5*x_6+x_3*x_5*x_6+x_6^2
 ] with indeterminate degrees [ 1, 1, 1, 2, 2, 3, 4 ]
```

The command `CohomologicalData(G,n)` prints complete information for the cohomology ring $H^*(G, Z_2)$ of a 2-group G provided that the integer n is at least the maximal degree of a relator in a minimal set of relators for the ring. Groebner basis routines from SINGULAR are called involved in the example.

The following example produces complete information on the Steenrod algebra of group number 8 in GAP's library of groups of order 32. (This example take over 2 hours to run. Most other groups of order 32 run significantly quicker.)

```
─────────────────────────────── Example ───────────────────────────────
 gap> CohomologicalData(SmallGroup(32,8),12);

 Group number: 8
 Group description: C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2)
```

```
Cohomology generators
Degree 1: a, b
Degree 2: c, d
Degree 3: e
Degree 5: f, g
Degree 6: h
Degree 8: p

Cohomology relations
1: f^2
2: c*h+e*f
3: c*f
4: b*h+c*g
5: b*e+c*d
6: a*h
7: a*g
8: a*f+b*f
9: a*e+c^2
10: a*c
11: a*b
12: a^2
13: d*e*h+e^2*g+f*h
14: d^2*h+d*e*f+d*e*g+f*g
15: c^2*d+b*f
16: b*c*g+e*f
17: b*c*d+c*e
18: b^2*g+d*f
19: b^2*c+c^2
20: b^3+a*d
21: c*d^2*e+c*d*g+d^2*f+e*h
22: c*d^3+d*e^2+d*h+e*f+e*g
23: b^2*d^2+c*d^2+b*f+e^2
24: b^3*d
25: d^3*e^2+d^2*e*f+c^2*p+h^2
26: d^4*e+b*c*p+e^2*g+g*h
27: d^5+b*d^2*g+b^2*p+f*g+g^2

Poincare series
(x^5+x^2+1)/(x^8-2*x^7+2*x^6-2*x^5+2*x^4-2*x^3+2*x^2-2*x+1)

Steenrod squares
Sq^1(c)=0
Sq^1(d)=b*b*b+d*b
Sq^1(e)=c*b*b
Sq^2(e)=e*d+f
Sq^1(f)=c*d*b*b+d*d*b*b
Sq^2(f)=g*b*b
Sq^4(f)=p*a
Sq^1(g)=d*d*d+g*b
Sq^2(g)=0
Sq^4(g)=c*d*d*d*b+g*d*b*b+g*d*d+p*a+p*b
Sq^1(h)=c*d*d*b+e*d*d
```

```
Sq^2(h)=d*d*d*b*b+c*d*d*d+g*c*b
Sq^4(h)=d*d*d*d*b*b+g*e*d+p*c
Sq^1(p)=c*d*d*d*b
Sq^2(p)=d*d*d*d*b*b+c*d*d*d*d
Sq^4(p)=d*d*d*d*d*b*b+d*d*d*d*d*d*d+g*d*d*d*b+g*g*d+p*d*d
```

8.2 Steenrod operations on the classifying space of a finite p-group

The following example constructs the first eight degrees of the mod-3 cohomology ring $H^*(G, \mathbb{Z}_3)$ for the group G number 4 in GAP's library of groups of order 81. It determines a minimal set of ring generators lying in degree ≤ 8 and it evaluates the Bockstein operator on these generators. Steenrod powers for $p \geq 3$ are not implemented as no efficient method of implementation is known.

```
                            Example
gap> G:=SmallGroup(81,4);;
gap> A:=ModPSteenrodAlgebra(G,8);;
gap> List(ModPRingGenerators(A),x->Bockstein(A,x));
[ 0*v.1, 0*v.1, v.5, 0*v.1, (Z(3))*v.7+v.8+(Z(3))*v.9 ]
```

Chapter 9

Bredon homology

9.1 Davis complex

The following example computes the Bredon homology

$\underline{H}_0(W, \mathscr{R}) = \mathbb{Z}^{21}$

for the infinite Coxeter group W associated to the Dynkin diagram shown in the computation, with coefficients in the complex representation ring.

```
 ───────────────────────────── Example ─────────────────────────────
 gap> D:=[[1,[2,3]],[2,[3,3]],[3,[4,3]],[4,[5,6]]];;
 gap> CoxeterDiagramDisplay(D);
```

```
 ───────────────────────────── Example ─────────────────────────────
 gap> C:=DavisComplex(D);;
 gap> D:=TensorWithComplexRepresentationRing(C);;
 gap> Homology(D,0);
 [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

9.2 Arithmetic groups

The following example computes the Bredon homology

$\underline{H}_0(SL_2(\mathscr{O}_{-3}), \mathscr{R}) = \mathbb{Z}_2 \oplus \mathbb{Z}^9$

$\underline{H}_1(SL_2(\mathscr{O}_{-3}), \mathscr{R}) = \mathbb{Z}$

for $\mathscr{O}_{-3}$ the ring of integers of the number field $\mathbb{Q}(\sqrt{-3})$, and $\mathscr{R}$ the complex reflection ring.

```
 ───────────────────────────── Example ─────────────────────────────
 gap> R:=ContractibleGcomplex("SL(2,O-3)");;
 gap> IsRigid(R);
 false
 gap> S:=BaryCentricSubdivision(R);;
 gap> IsRigid(S);
 true
 gap> C:=TensorWithComplexRepresentationRing(S);;
 gap> Homology(C,0);
 [ 2, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
 gap> Homology(C,1);
```

```
[ 0 ]
```

9.3 Crystallographic groups

The following example computes the Bredon homology
$$\underline{H}_0(G,\mathscr{R}) = \mathbb{Z}^{17}$$
for G the second crystallographic group of dimension 4 in GAP's library of crystallographic groups, and for $\mathscr{R}$ the Burnside ring.

```
                            Example
gap> G:=SpaceGroup(4,2);;
gap> gens:=GeneratorsOfGroup(G);;
gap> B:=CrystGFullBasis(G);;
gap> R:=CrystGcomplex(gens,B,1);;
gap> IsRigid(R);
false
gap> S:=CrystGcomplex(gens,B,0);;
gap> IsRigid(S);
true
gap> D:=TensorWithBurnsideRing(S);;
gap> Homology(D,0);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

Chapter 10

Chain Complexes

HAP uses implementations of chain complexes of free $\mathbb{K}$-modules for each of the rings $\mathbb{K} = \mathbb{Z}$, $\mathbb{K} = \mathbb{Q}$, $\mathbb{K} = \mathbb{F}_p$ with p a prime number, $\mathbb{K} = \mathbb{Z}G$, $\mathbb{K} = \mathbb{F}_pG$ with G a group. The implemented chain complexes have the form

$$C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0 .$$

Such a complex is said to have *length n* and the rank of the free $\mathbb{K}$-module C_k is referred to as the *dimenion* of the complex in degree k.

For the case $\mathbb{K} = \mathbb{Z}G$ (resp. $\mathbb{K} = \mathbb{F}_pG$) the main focus is on free chain complexes that are exact at each degree k, i.e. $\mathrm{im}(d_{k+1}) = \ker(d_k)$, for $0 < k < n$ and with $C_0/\mathrm{im}(d_1) \cong \mathbb{Z}$ (resp. $C_0/\mathrm{im}(d_1) \cong \mathbb{F}_p$). We refer to such a chain complex as a *resolution of length n* even though d_n will typically not be injective. More correct terminology would refer to such a chain complex as the first n degrees of a free resolution.

The following sections illustrate some constructions of chain complexes. Constructions for resolutions are described in the next chapter 11.

10.1 Chain complex of a simplicial complex and simplicial pair

The following example constructs the Quillen simplicial complex $Q = \mathscr{A}_p(G)$ for $p = 2$ and $G = A_8$; this is the order complex of the poset of non-trivial elementary 2-subgroups of G. The chain complex $C_* = C_*(Q)$ is then computed and seen to have the same number of free generators as Q has simplices. (To ensure indexing of subcomplexes is consistent with that of the large complex it is best to work with vertices represented as integers.)

```
------------------------------ Example ------------------------------
  gap> Q:=QuillenComplex(AlternatingGroup(8),2);
  Simplicial complex of dimension 3.

  gap> C:=ChainComplex(Q);
  Chain complex of length 3 in characteristic 0 .

  gap> Size(Q);
  55015
  gap> Size(C);
  55015
```

Next the simplicial complex Q is converted to one whose vertices are represented by integers and a contactible subcomplex $L < Q$ is computed. The chain complex $D_* = C_*(Q,L)$ of the simplicial pair (Q,L) is constructed and seen to have the correct size.

```
———————————————————————— Example ————————————————————————
  gap> Q:=IntegerSimplicialComplex(Q);
  Simplicial complex of dimension 3.

  gap> L:=ContractibleSubcomplex(Q);
  Simplicial complex of dimension 3.

  gap> D:=ChainComplexOfPair(Q,L);
  Chain complex of length 3 in characteristic 0 .

  gap> Size(D)=Size(Q)-Size(L);
  true
  gap> Size(D);
  670
  gap>
```

The next commands produce a smalled chain complex B_* chain homotopy equivalent to D_* and compute the homology $H_k(Q,\mathbb{Z}) \cong H_k(B_*)$ for $k = 1,2,3$.

```
———————————————————————— Example ————————————————————————
  gap> B:=ContractedComplex(D);
  Chain complex of length 3 in characteristic 0 .

  gap> Size(B);
  64
  gap> Homology(B,1);
  [  ]
  gap> Homology(B,2);
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
  gap> Homology(B,3);
  [  ]
```

10.2   Chain complex of a cubical complex and cubical pair

The following example reads in the digital image
   as a 2-dimensional pure cubical complex M and constructs the chain complex $C_* = C_*(M)$.

```
———————————————————————— Example ————————————————————————
  gap> K:=ReadImageAsPureCubicalComplex(file,400);
  Pure cubical complex of dimension 2.

  gap> C:=ChainComplex(K);
  Chain complex of length 2 in characteristic 0 .

  gap> Size(C);
```

```
   173243
```

Next an acyclic pure cubical subcomplex $L < M$ is computed and the chain complex $D_* = C_*(M, L)$ of the pair is constructed.

```
                            ——————————— Example ———————————
  gap> L:=AcyclicSubcomplexOfPureCubicalComplex(K);
  Pure cubical complex of dimension 2.

  gap> D:=ChainComplexOfPair(K,L);
  Chain complex of length 2 in characteristic 0 .

  gap> Size(D);
  618
```

Finally the chain complex D_* is simplified to a homotopy equivalent chain complex B_* and the homology $H_1(M, \mathbb{Z}) \cong H_1(B_*)$ is computed.

```
                            ——————————— Example ———————————
  gap> B:=ContractedComplex(D);
  Chain complex of length 2 in characteristic 0 .

  gap> Size(B);
  20
  gap> Homology(B,1);
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

10.3   Chain complex of a regular CW-complex

The next example constructs a 15-dimensional regular CW-complex Y that is homotopy equivalent to the 2-dimensional torus.

```
                            ——————————— Example ———————————
  gap> Circle:=PureCubicalComplex([[1,1,1,1,1],[1,1,0,1,1],[1,1,1,1,1]]);
  Pure cubical complex of dimension 2.

  gap> Torus:=DirectProductOfPureCubicalComplexes(Circle,Circle);
  Pure cubical complex of dimension 4.

  gap> CTorus:=CechComplexOfPureCubicalComplex(Torus);
  Simplicial complex of dimension 15.

  gap> Y:=RegularCWComplex(CTorus);
  Regular CW-complex of dimension 15
```

Next the cellular chain complex $C_* = C_*(Y)$ is constructed. Also, a minimally generated chain complex $D_* = C_*(Y')$ of a non-regular CW-complex $Y' \simeq Y$ is constructed.

```
                              ───── Example ─────
  gap> C:=ChainComplexOfRegularCWComplex(Y);
  Chain complex of length 15 in characteristic 0 .

  gap> Size(C);
  1172776

  gap> D:=ChainComplex(Y);
  Chain complex of length 15 in characteristic 0 .

  gap> Size(D);
  4
```

10.4   Chain Maps of simplicial and regular CW maps

The next example realizes the complement of the first prime knot on 11 crossings as a pure permutahedral complex. The complement is converted to a regular CW-complex Y and the boundary inclusion $f: \partial Y \hookrightarrow Y$ is constructed as a map of regular CW-complexes. Then the induced chain map $F: C_*(\partial Y) \hookrightarrow C_*(Y)$ is constructed. Finally the homology homomorphism $H_1(F): H_1(C_*(\partial Y)) \to H_1(C_*(Y))$ is computed.

```
                              ───── Example ─────
  gap> K:=PurePermutahedralKnot(11,1);;
  gap> M:=PureComplexComplement(K);
  Pure permutahedral complex of dimension 3.

  gap> Y:=RegularCWComplex(M);
  Regular CW-complex of dimension 3

  gap> f:=BoundaryMap(Y);
  Map of regular CW-complexes

  gap> F:=ChainMap(f);
  Chain Map between complexes of length 2 .

  gap> H:=Homology(F,1);
  [ g1, g2 ] -> [ g1^-1, g1^-1 ]

  gap> Kernel(H);
  Pcp-group with orders [ 0 ]
```

The command ChainMap(f) can be used to construct the chain map $C_*(K) \to C_*(K')$ induced by a map $f: K \to K'$ of simplicial complexes.

10.5   Constructions for chain complexes

It is straightforward to implement basic constructions on chain complexes. A few constructions are illustrated in the following example.

```
───────────────────────── Example ─────────────────────────
gap> res:=ResolutionFiniteGroup(SymmetricGroup(5),5);;
gap> C:=TensorWithIntegers(res);
Chain complex of length 5 in characteristic 0 .

gap> D:=ContractedComplex(C);#A chain homotopic complex
Chain complex of length 5 in characteristic 0 .
gap> List([0..5],C!.dimension);
[ 1, 4, 10, 20, 35, 56 ]
gap> List([0..5],D!.dimension);
[ 1, 1, 2, 4, 6, 38 ]

gap> CxC:=TensorProduct(C,C);
Chain complex of length 10 in characteristic 0 .

gap> SC:=SuspendedChainComplex(C);
Chain complex of length 6 in characteristic 0 .

gap> RC:=ReducedSuspendedChainComplex(C);
Chain complex of length 6 in characteristic 0 .

gap> PC:=PathObjectForChainComplex(C);
Chain complex of length 5 in characteristic 0 .

gap> dualC:=HomToIntegers(C);
Cochain complex of length 5 in characteristic 0 .

gap> Cxp:=TensorWithIntegersModP(C,5);
Chain complex of length 5 in characteristic 5 .

gap> CxQ:=TensorWithRationals(C); #The quirky -1/2 denotes rationals
Chain complex of length 5 in characteristic -1/2 .
```

10.6 Filtered chain complexes

A sequence of inclusions of chain complexes $C_{0,*} \leq C_{1,*} \leq \cdots \leq C_{T-1,*} \leq C_{T,*}$ in which the preferred basis of $C_{k-1,\ell}$ is the beginning of the preferred basis of $C_{k,\ell}$ is referred to as a *filtered chain complex*. Filtered chain complexes give rise to spectral sequences such as the *equivariant spectral sequence* of a $G - CW$-complex with subgroup $H < G$. A particular case is the Lyndon-Hochschild-Serre spectral sequence for the homology of a group extension $N \rightarrowtail G \twoheadrightarrow Q$ with $E^2_{p,q} = H_p(Q, H_q(N, \mathbb{Z}))$.

   The following commands construct the filtered chain complex underlying the Lyndon-Hochschild-Serre spectral sequence for the dihedral group $G = D_{32}$ of order 64 and its centre $N = Z(G)$.

```
───────────────────────── Example ─────────────────────────
gap> G:=DihedralGroup(64);;
gap> N:=Center(G);;
gap> R:=ResolutionNormalSeries([G,N],3);;
gap> C:=FilteredTensorWithIntegersModP(R,2);
Chain complex of length 3 in characteristic 2 .
```

The differentials $d_{p,q}^r$ in a given page E^r of the spectral sequence arise from the induced homology homomorphisms $\iota_\ell^{s,t}:H_\ell(C_{s,*}) \to H_\ell(C_{t,*})$ for $s \leq t$. Textbooks traditionally picture the differential in E^r as an array of sloping arrows with non-zero groups $E_{p,q}^r \neq 0$ represented by dots. An alternative representation of this information is as a barcode (of the sort used in Topological Data Analysis). The homomorphisms $\iota_2^{*,*}$ in the example, with coefficients converted to mod 2, are pictured by the bar code

which was produced by the following commands.

```
————————————————————————— Example —————————————————————————
gap> p:=2;;k:=2;;
gap> P:=PersistentHomologyOfFilteredChainComplex(C,k,p);;
gap> BarCodeDisplay(P);
```

10.7 Sparse chain complexes

Boundary homomorphisms in all of the above examples of chain complexes are represented by matrices. In cases where the matrices are large and have many zero entries it is better to use sparse matrices.

The following commands demonstrate the conversion of the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

to sparse form, and vice-versa.

```
————————————————————————— Example —————————————————————————
gap> A:=[[0,2,0],[-3,0,0],[0,0,4]];;
gap> S:=SparseMat(A);
Sparse matrix with 3 rows and 3 columns in characteristic 0

gap> NamesOfComponents(S);
[ "mat", "characteristic", "rows", "cols" ]
gap> S!.mat;
[ [ [ 2, 2 ] ], [ [ 1, -3 ] ], [ [ 3, 4 ] ] ]

gap> B:=SparseMattoMat(S);
[ [ 0, 2, 0 ], [ -3, 0, 0 ], [ 0, 0, 4 ] ]
```

To illustrate the use of sparse chain complexes we consider the data points represented in the following digital image.

The following commands read in this image as a 2-dimensional pure cubical complex and store the Euclidean coordinates of the black pixels in a list. Then 200 points are selected at random from this list and used to construct a 200×200 symmetric matrix S whose entries are the Euclidean distance between the sample data points.

```
————————————————————————— Example —————————————————————————
gap> file:=HapFile("data500.png");;
gap> M:=ReadImageAsPureCubicalComplex(file,400);;
gap> A:=M!.binaryArray;;
gap> data:=[];;
gap> for i in [1..Length(A)] do
```

```
> for j in [1..Length(A[1])] do
> if A[i][j]=1 then Add(data,[i,j]); fi;
> od;
> od;
gap> sample:=List([1..200],i->Random(data));;
gap> S:=VectorsToSymmetricMatrix(sample,EuclideanApproximatedMetric);;
```

The symmetric distance matrix *S* is next converted to a filtered chain complex arising from a filtered simplicial complex (using the standard *persistent homology* pipeline).

———————————— Example ————————————
```
gap> G:=SymmetricMatrixToFilteredGraph(S,10,100);;
#Filtration length T=10, distances greater than 100 discarded.
gap> N:=SimplicialNerveOfFilteredGraph(G,2);;
gap> C:=SparseFilteredChainComplexOfFilteredSimplicialComplex(N);;
Filtered sparse chain complex of length 2 in characteristic 0 .
```

Next, the induced homology homomorphisms in degrees 1 and 2, with rational coefficients, are computed and displayed a barcodes.

———————————— Example ————————————
```
gap> P0:=PersistentHomologyOfFilteredSparseChainComplex(C,0);;
gap> P1:=PersistentHomologyOfFilteredSparseChainComplex(C,1);;
gap> BarCodeCompactDisplay(P0);
```

———————————— Example ————————————
```
gap> BarCodeCompactDisplay(P1);
```

The barcodes are consistent with the data points having been sampled from a space with the homotopy type of an annulus.

Chapter 11

Resolutions

There is a range of functions in HAP that input a group G, integer n, and attempt to return the first n terms of a free $\mathbb{Z}G$-resolution R_* of the trivial module $\mathbb{Z}$. In some cases an explicit contracting homotopy is provided on the resolution. The function Size(R) returns a list whose kth term is the sum of the lengths of the boundaries of the generators in degree k.

11.1 Resolutions for small finite groups

The following uses discrete Morse theory to construct a resolution.

```
————————————— Example —————————————
gap> G:=SymmetricGroup(6);; n:=6;;
gap> R:=ResolutionFiniteGroup(G,n);
Resolution of length 6 in characteristic 0 for Group([ (1,2), (1,2,3,4,5,6)
 ]) .

gap> Size(R);
[ 10, 58, 186, 452, 906, 1436 ]
```

11.2 Resolutions for very small finite groups

The following uses linear algebra over $\mathbb{Z}$ to construct a resolution.

```
————————————— Example —————————————
gap> Q:=QuaternionGroup(128);;
gap> R:=ResolutionSmallGroup(Q,20);
Resolution of length 20 in characteristic 0 for <pc group of size 128 with
2 generators> .
No contracting homotopy available.

gap> Size(R);
[ 4, 42, 8, 128, 4, 42, 8, 128, 4, 42, 8, 128, 4, 42, 8, 128, 4, 42, 8, 128 ]
```

The suspicion that this resolution R_* is periodic of period 4 can be verified by constructing the chain complex $C_* = R_* \otimes_{\mathbb{Z}} \mathbb{Z}G$ and verifying that boundary matrices repeat with period 4.

A second example of a periodic resolution, for the Dihedral group $D_{2k+1} = \langle x, y \mid x^2 = xy^k x^{-1} y^{-k-1} \rangle$ of order $2k+2$ in the case $k = 1$, is constructed and verified for periodicity in the next example.

```
———————————————————— Example ————————————————————
gap> R:=ResolutionSmallGroup(D,15);;
gap> Size(R);
[ 4, 7, 8, 6, 4, 8, 8, 6, 4, 8, 8, 6, 4, 8, 8 ]
gap> C:=TensorWithIntegersOverSubgroup(R,Group(One(D)));;
gap> n:=4;;BoundaryMatrix(C,n)=BoundaryMatrix(C,n+4);
true
gap> n:=5;;BoundaryMatrix(C,n)=BoundaryMatrix(C,n+4);
true
gap> n:=6;;BoundaryMatrix(C,n)=BoundaryMatrix(C,n+4);
true
gap> n:=7;;BoundaryMatrix(C,n)=BoundaryMatrix(C,n+4);
true
gap> n:=8;;BoundaryMatrix(C,n)=BoundaryMatrix(C,n+4);
true
```

This periodic resolution for D_3 can be found in a paper by R. Swan [Swa60]. The resolution was proved for arbitrary D_{2k+1} by Irina Kholodna [Kho01] (Corollary 5.5) and is the cellular chain complex of the universal cover of a CW-complex X with two cells in dimensions $1, 2$ mod 4 and one cell in dimensions $0, 3$ mod 4. The 2-skelecton is the 2-complex for the given presentation of D_{2k+1} and an attaching map for the 3-cell is represented as follows.

A slightly different periodic resolution for D_{2k+1} has been obtain more recently by FEA Johnson [Joh16]. Johnson's resolution has two free generators in each degree. Interestingly, running the following code for many values of $k > 1$ seems to produce a periodic resolution with two free generators in each degree for most values of k.

```
———————————————————— Example ————————————————————
gap> k:=20;;rels:=[x^2,x*y^k*x^-1*y^(-1-k)];;D:=F/rels;;
gap> R:=ResolutionSmallGroup(D,7);;
gap> List([0..7],R!.dimension);
[ 1, 2, 2, 2, 2, 2, 2, 2 ]
```

11.3   Resolutions for finite groups acting on orbit polytopes

The following uses Polymake convex hull computations and homological perturbation theory to construct a resolution.

```
———————————————————— Example ————————————————————
gap> G:=SignedPermutationGroup(5);;
gap> StructureDescription(G);
"C2 x ((C2 x C2 x C2 x C2) : S5)"

gap> v:=[1,2,3,4,5];;  #The resolution depends on the choice of vector.
gap> P:=PolytopalComplex(G,[1,2,3,4,5]);
Non-free resolution in characteristic 0 for <matrix group of size 3840 with
9 generators> .
No contracting homotopy available.
```

```
gap> R:=FreeGResolution(P,6);
Resolution of length 5 in characteristic 0 for <matrix group of size
3840 with 9 generators> .
No contracting homotopy available.
gap> Size(R);
[ 10, 60, 214, 694, 6247, 273600 ]
```

The convex polytope $P_G(v) = \text{Convex Hull}\{g \cdot v \mid g \in G\}$ used in the resolution depends on the choice of vector $v \in \mathbb{R}^n$. Two such polytopes for the alternating group $G = A_4$ acting on $\mathbb{R}^4$ can be visualized as follows.

```
                          ─ Example ─
gap> G:=AlternatingGroup(4);;
gap> OrbitPolytope(G,[1,2,3,4],["VISUAL"]);
gap> OrbitPolytope(G,[1,1,3,4],["VISUAL"]);

gap> P1:=PolytopalComplex(G,[1,2,3,4]);;
gap> P2:=PolytopalComplex(G,[1,1,3,4]);;
gap> R1:=FreeGResolution(P1,20);;
gap> R2:=FreeGResolution(P2,20);;
gap> Size(R1);
[ 6, 11, 32, 24, 36, 60, 65, 102, 116, 168, 172, 248, 323, 628, 650, 1093,
  1107, 2456, 2344, 6115 ]
gap> Size(R2);
[ 4, 11, 20, 24, 36, 60, 65, 102, 116, 168, 172, 248, 323, 628, 650, 1093,
  1107, 2456, 2344, 6115 ]
```

11.4 Minimal resolutions for finite p-groups over $\mathbb{F}_p$

The following uses linear algebra to construct a minimal free $\mathbb{F}_p G$-resolution of the trivial module $\mathbb{F}$.

```
                          ─ Example ─
gap> P:=SylowSubgroup(MathieuGroup(12),2);;
gap> R:=ResolutionPrimePowerGroup(P,20);
Resolution of length 20 in characteristic 2 for Group(
[ (2,8,4,12)(3,11,7,9), (2,3)(4,7)(6,10)(9,11), (3,7)(6,10)(8,11)(9,12),
  (1,10)(3,7)(5,6)(8,12), (2,4)(3,7)(8,12)(9,11), (1,5)(6,10)(8,12)(9,11)
 ]) .

gap> Size(R);
[ 6, 62, 282, 740, 1810, 3518, 6440, 10600, 17040, 24162, 34774, 49874,
  62416, 81780, 106406, 145368, 172282, 208926, 262938, 320558 ]
```

The resolution has the minimum number of generators possible in each degree and can be used to guess a formula for the Poincare series

$P(x) = \Sigma_{k \geq 0} \dim_{\mathbb{F}_p} H^k(G, \mathbb{F}_p) x^k.$

The guess is certainly correct for the coefficients of x^k for $k \leq 20$ and can be used to guess the dimension of say $H^{2000}(G, \mathbb{F}_p)$.

Most likely $\dim_{\mathbb{F}_2} H^{2000}(G, \mathbb{F}_2) = 2001000$.

```
                          _____ Example _____
gap> P:=PoincareSeries(R,20);
(1)/(-x_1^3+3*x_1^2-3*x_1+1)

gap> ExpansionOfRationalFunction(P,2000)[2000];
2001000
```

11.5   Resolutions for abelian groups

The following uses the formula for the tensor product of chain complexes to construct a resolution.

```
                          _____ Example _____
gap> A:=AbelianPcpGroup([2,4,8,0,0]);;
gap> StructureDescription(A);
"Z x Z x C8 x C4 x C2"

gap> R:=ResolutionAbelianGroup(A,10);
Resolution of length 10 in characteristic 0 for Pcp-group with orders
[ 2, 4, 8, 0, 0 ] .

gap> Size(R);
[ 14, 90, 296, 680, 1256, 2024, 2984, 4136, 5480, 7016 ]
```

11.6   Resolutions for nilpotent groups

The following uses the NQ package to express the free nilpotent group of class 3 on three generators as a Pcp group G, and then uses homological perturbation on the lower central series to construct a resolution. The resolution is used to exhibit 2-torsion in $H_4(G, \mathbb{Z})$.

```
                          _____ Example _____
gap> F:=FreeGroup(3);;
gap> G:=Image(NqEpimorphismNilpotentQuotient(F,3));;
gap> R:=ResolutionNilpotentGroup(G,5);
Resolution of length 5 in characteristic 0 for Pcp-group with orders
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] .

gap> Size(R);
[ 28, 377, 2377, 9369, 25850 ]

gap> Homology(TensorWithIntegers(R),4);
[ 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

The following uses homological perturbation on the lower central series to construct a resolution for the Sylow 2-subgroup $P = Syl_2(M_{12})$ of the Mathieu simple group M_{12}.

```
—————————————————————— Example ——————————————————————
gap> G:=MathieuGroup(12);;
gap> P:=SylowSubgroup(G,2);;
gap> StructureDescription(P);
"((C4 x C4) : C2) : C2"

gap> R:=ResolutionNilpotentGroup(P,9);
Resolution of length 9 in characteristic
0 for <permutation group with 279 generators> .

gap> Size(R);
[ 12, 80, 310, 939, 2556, 6768, 19302, 61786, 237068 ]
```

11.7 Resolutions for groups with subnormal series

The following uses homological perturbation on a subnormal series to construct a resolution for the Sylow 2-subgroup $P = Syl_2(M_{12})$ of the Mathieu simple group M_{12}.

```
—————————————————————— Example ——————————————————————
gap> P:=SylowSubgroup(MathieuGroup(12),2);;
gap> sn:=ElementaryAbelianSeries(P);;
gap> R:=ResolutionSubnormalSeries(sn,9);
Resolution of length 9 in characteristic
0 for <permutation group with 64 generators> .

gap> Size(R);
[ 12, 78, 288, 812, 1950, 4256, 8837, 18230, 39120 ]
```

11.8 Resolutions for groups with normal series

The following uses homological perturbation on a normal series to construct a resolution for the Sylow 2-subgroup $P = Syl_2(M_{12})$ of the Mathieu simple group M_{12}.

```
—————————————————————— Example ——————————————————————
gap> P:=SylowSubgroup(MathieuGroup(12),2);;
gap> P1:=EfficientNormalSubgroups(P)[1];;
gap> P2:=Intersection(DerivedSubgroup(P),P1);;
gap> P3:=Group(One(P));;
gap> R:=ResolutionNormalSeries([P,P1,P2,P3],9);
Resolution of length 9 in characteristic
0 for <permutation group with 64 generators> .

gap> Size(R);
[ 10, 60, 200, 532, 1238, 2804, 6338, 15528, 40649 ]
```

11.9 Resolutions for polycyclic (almost) crystallographic groups

The following uses the Polycyclic package and homological perturbation to construct a resolution for the crystallographic group G:=SpaceGroup(3,165).

```
                         ───── Example ─────
  gap> G:=SpaceGroup(3,165);;
  gap> G:=Image(IsomorphismPcpGroup(G));;
  gap> R:=ResolutionAlmostCrystalGroup(G,20);
  Resolution of length 20 in characteristic 0 for Pcp-group with orders
  [ 3, 2, 0, 0, 0 ] .

  gap> Size(R);
  [ 10, 49, 117, 195, 273, 351, 429, 507, 585, 663, 741, 819, 897, 975, 1053,
    1131, 1209, 1287, 1365, 1443 ]
```

The following constructs a resolution for an almost crystallographic Pcp group *G*. The final commands establish that *G* is not isomorphic to a crystallographic group.

```
                         ───── Example ─────
  gap> G:=AlmostCrystallographicPcpGroup( 4, 50, [ 1, -4, 1, 2 ] );;
  gap> R:=ResolutionAlmostCrystalGroup(G,20);
  Resolution of length 20 in characteristic 0 for Pcp-group with orders
  [ 4, 0, 0, 0, 0 ] .

  gap> Size(R);
  [ 10, 53, 137, 207, 223, 223, 223, 223, 223, 223, 223, 223, 223, 223, 223,
    223, 223, 223, 223, 223 ]


  gap> T:=Kernel(NaturalHomomorphismOnHolonomyGroup(G));;
  gap> IsAbelian(T);
  false
```

11.10 Resolutions for Bieberbach groups

The following constructs a resolution for the Bieberbach group G=SpaceGroup(3,165) by using convex hull algorithms to construct a Dirichlet domain for its free action on Euclidean space $\mathbb{R}^3$. By construction the resolution is trivial in degrees ≥ 3.

```
                         ───── Example ─────
  gap> G:=SpaceGroup(3,165);;
  gap> R:=ResolutionBieberbachGroup(G);
  Resolution of length 4 in characteristic
  0 for SpaceGroupOnRightBBNWZ( 3, 6, 1, 1, 4 ) .
  No contracting homotopy available.

  gap> Size(R);
  [ 10, 18, 8, 0 ]
```

The fundamental domain constructed for the above resolution can be visualized using the following commands.

```
────── Example ──────
gap> F:=FundamentalDomainBieberbachGroup(G);
<polymake object>
gap> Display(F);
```

A different fundamental domain and resolution for *G* can be obtained by changing the choice of vector $v \in \mathbb{R}^3$ in the definition of the Dirichlet domain

$$D(v) = \{x \in \mathbb{R}^3 \mid ||x - v|| \leq ||x - g.v|| \text{ for all } g \in G\}.$$

```
────── Example ──────
gap> R:=ResolutionBieberbachGroup(G,[1/2,1/2,1/2]);
Resolution of length 4 in characteristic
0 for SpaceGroupOnRightBBNWZ( 3, 6, 1, 1, 4 ) .
No contracting homotopy available.

gap> Size(R);
[ 28, 42, 16, 0 ]

gap> F:=FundamentalDomainBieberbachGroup(G);
<polymake object>
gap> Display(F);
```

A higher dimensional example is handled in the next session. A list of the 62 7-dimensional Hantze-Wendt Bieberbach groups is loaded and a resolution is computed for the first group in the list.

```
────── Example ──────
gap> file:=HapFile("HW-7dim.txt");;
gap> Read(file);
gap> G:=HWO7Gr[1];
<matrix group with 7 generators>

gap> R:=ResolutionBieberbachGroup(G);
Resolution of length 8 in characteristic 0 for <matrix group with
7 generators> .
No contracting homotopy available.

gap> Size(R);
[ 284, 1512, 3780, 4480, 2520, 840, 84, 0 ]
```

The homological perturbation techniques needed to extend this method to crystallographic groups acting non-freely on $\mathbb{R}^n$ has not yet been implemenyed. This is on the TO-DO list.

11.11 Resolutions for arbitrary crystallographic groups

An implementation of the above method for Bieberbach groups is also available for arbitrary crystallographic groups. The following example constructs a resolution for the group `G:=SpaceGroupIT(3,227)`.

```
━━━━━━━━━━━━━━━━━━━━━━ Example ━━━━━━━━━━━━━━━━━━━━━━
gap> G:=SpaceGroupIT(3,227);;
gap> R:=ResolutionSpaceGroup(G,11);
Resolution of length 11 in characteristic 0 for <matrix group with
8 generators> .
No contracting homotopy available.

gap> Size(R);
[ 38, 246, 456, 644, 980, 1427, 2141, 2957, 3993, 4911, 6179 ]
```

11.12 Resolutions for crystallographic groups admitting cubical fundamental domain

The following uses subdivision techniques to construct a resolution for the Bieberbach group
G:=SpaceGroup(4,122). The resolution is endowed with a contracting homotopy.

```
━━━━━━━━━━━━━━━━━━━━━━ Example ━━━━━━━━━━━━━━━━━━━━━━
gap> G:=SpaceGroup(4,122);;
gap> R:=ResolutionCubicalCrystGroup(G,20);
Resolution of length 20 in characteristic 0 for <matrix group with
6 generators> .

gap> Size(R);
[ 8, 24, 24, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

Subdivision and homological perturbation are used to construct the following resolution (with contracting homotopy) for a crystallographic group with non-free action.

```
━━━━━━━━━━━━━━━━━━━━━━ Example ━━━━━━━━━━━━━━━━━━━━━━
gap> G:=SpaceGroup(4,1100);;
gap> R:=ResolutionCubicalCrystGroup(G,20);
Resolution of length 20 in characteristic 0 for <matrix group with
8 generators> .

gap> Size(R);
[ 40, 215, 522, 738, 962, 1198, 1466, 1734, 2034, 2334, 2666, 2998, 3362,
  3726, 4122, 4518, 4946, 5374, 5834, 6294 ]
```

11.13 Resolutions for Coxeter groups

The following session constructs the Coxeter diagram for the Coxeter group $B = B_7$ of order 645120.
A resolution for G is then computed.

```
━━━━━━━━━━━━━━━━━━━━━━ Example ━━━━━━━━━━━━━━━━━━━━━━
gap> D:=[[1,[2,3]],[2,[3,3]],[3,[4,3]],[4,[5,3]],[5,[6,3]],[6,[7,4]]];;
gap> CoxeterDiagramDisplay(D);;
```

```
────────── Example ──────────
gap> R:=ResolutionCoxeterGroup(D,5);
Resolution of length 5 in characteristic
0 for <permutation group of size 645120 with 7 generators> .
No contracting homotopy available.

gap> Size(R);
[ 14, 112, 492, 1604, 5048 ]
```

The routine extension of this method to infinite Coxeter groups is on the TO-DO list.

11.14 Resolutions for Artin groups

The following session constructs a resolution for the infinite Artin group G associated to the Coxeter group B_7. Exactness of the resolution depends on the solution to the $K(\pi,1)$ Conjecture for Artin groups of spherical type.

```
────────── Example ──────────
gap> R:=ResolutionArtinGroup(D,8);
Resolution of length 8 in characteristic 0 for <fp group on the generators
[ f1, f2, f3, f4, f5, f6, f7 ]> .
No contracting homotopy available.

gap> Size(R);
[ 14, 98, 310, 610, 918, 1326, 2186, 0 ]
```

11.15 Resolutions for $G = SL_2(\mathbb{Z}[1/m])$

The following uses homological perturbation to construct a resolution for $G = SL_2(\mathbb{Z}[1/6])$.

```
────────── Example ──────────
gap> R:=ResolutionSL2Z(6,10);
Resolution of length 10 in characteristic 0 for SL(2,Z[1/6]) .

gap> Size(R);
[ 44, 679, 6910, 21304, 24362, 48506, 43846, 90928, 86039, 196210 ]
```

11.16 Resolutions for selected groups $G = SL_2(\mathscr{O}(\mathbb{Q}(\sqrt{d}))$

The following uses finite "Voronoi complexes" and homological perturbation to construct a resolution for $G = SL_2(\mathscr{O}(\mathbb{Q}(\sqrt{-5}))$. The finite complexes were contributed independently by A. Rahm, M. Dutour-Scikiric and S. Schoenenbeck and are stored in the folder ~pkg/Hap1.v/lib/Perturbations/Gcomplexes.

```
────────── Example ──────────
gap> R:=ResolutionSL2QuadraticIntegers(-5,10);
Resolution of length 10 in characteristic 0 for matrix group .
No contracting homotopy available.
```

```
gap> Size(R);
[ 22, 114, 120, 200, 146, 156, 136, 254, 168, 170 ]
```

11.17 Resolutions for selected groups $G = PSL_2(\mathscr{O}(\mathbb{Q}(\sqrt{d}))$

The following uses finite "Voronoi complexes" and homological perturbation to construct a resolution for $G = PSL_2(\mathscr{O}(\mathbb{Q}(\sqrt{-11}))$. The finite complexes were contributed independently by A. Rahm, M. Dutour-Scikiric and S. Schoenenbeck and are stored in the folder ~pkg/Hap1.v/lib/Perturbations/Gcomplexes.

———— Example ————
```
gap> R:=ResolutionPSL2QuadraticIntegers(-11,10);
Resolution of length 10 in characteristic 0 for PSL(2,O-11) .
No contracting homotopy available.

gap> Size(R);
[ 12, 59, 89, 107, 125, 230, 208, 270, 326, 515 ]
```

11.18 Resolutions for a few higher-dimensional arithmetic groups

The following uses finite "Voronoi complexes" and homological perturbation to construct a resolution for $G = PSL_4(\mathbb{Z})$. The finite complexes were contributed by M. Dutour-Scikiric and are stored in the folder ~pkg/Hap1.v/lib/Perturbations/Gcomplexes.

———— Example ————
```
gap>  V:=ContractibleGcomplex("PSL(4,Z)_d");
Non-free resolution in characteristic 0 for matrix group .
No contracting homotopy available.

gap> R:=FreeGResolution(V,5);
Resolution of length 5 in characteristic 0 for matrix group .
No contracting homotopy available.

gap> Size(R);
[ 18, 210, 1444, 26813 ]
```

11.19 Resolutions for finite-index subgroups

The next commands first construct the congruence subgroup $\Gamma_0(I)$ of index 144 in $SL_2(\mathscr{O}\mathbb{Q}(\sqrt{-2}))$ for the ideal I in $\mathscr{O}\mathbb{Q}(\sqrt{-2})$ generated by $4 + 5\sqrt{-2}$. The commands then compute a resolution for the congruence subgroup $G = \Gamma_0(I) \leq SL_2(\mathscr{O}\mathbb{Q}(\sqrt{-2}))$

———— Example ————
```
gap> Q:=QuadraticNumberField(-2);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,4+5*Sqrt(-2));;
```

```
gap> G:=HAP_CongruenceSubgroupGamma0(I);
<[group of 2x2 matrices in characteristic 0>
gap>
gap> IndexInSL2O(G);
144
gap> R:=ResolutionSL2QuadraticIntegers(-2,4,true);;
gap> S:=ResolutionFiniteSubgroup(R,G);
Resolution of length 4 in characteristic 0 for <matrix group with
290 generators> .

gap> Size(S);
[ 1152, 8496, 30960, 59616 ]
```

11.20   Simplifying resolutions

The next commands construct a resolution R_* for the symmetric group S_5 and convert it to a resolution S_* for the finite index subgroup $A_4 < S_5$. An heuristic algorithm is applied to S_* in the hope of obtaining a smaller resolution T_* for the alternating group A_4.

```
————————————————————— Example —————————————————————
gap> R:=ResolutionFiniteGroup(SymmetricGroup(5),5);;
gap> S:=ResolutionFiniteSubgroup(R,AlternatingGroup(4));
Resolution of length 5 in characteristic 0 for Alt( [ 1 .. 4 ] ) .

gap> Size(S);
[ 80, 380, 1000, 2040, 3400 ]
gap> T:=SimplifiedComplex(S);
Resolution of length 5 in characteristic 0 for Alt( [ 1 .. 4 ] ) .

gap> Size(T);
[ 4, 34, 22, 19, 196 ]
```

11.21   Resolutions for graphs of groups and for groups with aspherical presentations

The following example constructs a resolution for a finitely presented group whose presentation is known to have the property that its associated 2-complex is aspherical.

```
————————————————————— Example —————————————————————
gap> F:=FreeGroup(3);;x:=F.1;;y:=F.2;;z:=F.3;;
gap> rels:=[x*y*x*(y*x*y)^-1, y*z*y*(z*y*z)^-1, z*x*z*(x*z*x)^-1];;
gap> G:=F/rels;;
gap> R:=ResolutionAsphericalPresentation(G,10);
Resolution of length 10 in characteristic 0 for <fp group on the generators
[ f1, f2, f3 ]> .
No contracting homotopy available.

gap> Size(R);
```

```
[ 6, 18, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

The following commands create a resolution for a graph of groups corresponding to the amalgamated product $G = H *_A K$ where $H = S_5$ is the symmetric group of degree 5, $K = S_4$ is the symmetric group of degree 4 and the common subgroup is $A = S_3$.

```
——————————————————— Example ———————————————————
gap> S5:=SymmetricGroup(5);SetName(S5,"S5");;
Sym( [ 1 .. 5 ] )
gap> S4:=SymmetricGroup(4);SetName(S4,"S4");;
Sym( [ 1 .. 4 ] )
gap> A:=SymmetricGroup(3);SetName(A,"S3");;
Sym( [ 1 .. 3 ] )
gap> AS5:=GroupHomomorphismByFunction(A,S5,x->x);;
gap> AS4:=GroupHomomorphismByFunction(A,S4,x->x);;
gap> D:=[S5,S4,[AS5,AS4]];;
gap> GraphOfGroupsDisplay(D);;
```

```
——————————————————— Example ———————————————————
gap> R:=ResolutionGraphOfGroups(D,8);;
gap> Size(R);
[ 16, 68, 162, 302, 480, 627, 869, 1290 ]
```

11.22   Resolutions for $\mathbb{F}G$-modules

Let $\mathbb{F} = \mathbb{F}_p$ be the field of p elements and let M be some $\mathbb{F}G$-module. We might wish to construct a free $\mathbb{F}G$-resolution for M. We can handle this by constructing a short exact sequence

   $DM \rightarrowtail P \twoheadrightarrow M$

   in which P is free (or projective). Then any resolution of DM yields a resolution of M and we can represent DM as a submodule of P. We refer to DM as the *desuspension* of M. Consider for instance $G = Syl_2(GL(4,2))$ and $\mathbb{F} = \mathbb{F}_2$. The matrix group G acts via matrix multiplication on $M = \mathbb{F}^4$. The following example constructs a free $\mathbb{F}G$-resolution for M.

```
——————————————————— Example ———————————————————
gap> G:=GL(4,2);;
gap> S:=SylowSubgroup(G,2);;
gap> M:=GModuleByMats(GeneratorsOfGroup(S),GF(2));;
gap> DM:=DesuspensionMtxModule(M);;
gap> R:=ResolutionFpGModule(DM,20);
Resolution of length 20 in characteristic 2 for <matrix group of
size 64 with 3 generators> .

gap> List([0..20],R!.dimension);
[ 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136,
153, 171, 190, 210, 231, 253 ]
```

Chapter 12

Simplicial groups

12.1 Crossed modules

A *crossed module* consists of a homomorphism of groups $\partial\colon M \to G$ together with an action $(g,m) \mapsto {}^g m$ of G on M satisfying

1. $\partial({}^g m) = gmg^{-1}$

2. ${}^{\partial m} m' = mm'm^{-1}$

for $g \in G$, $m, m' \in M$.

A crossed module $\partial\colon M \to G$ is equivalent to a cat^1-group (H, s, t) (see 6.8) where $H = M \rtimes G$, $s(m, g) = (1, g)$, $t(m, g) = (1, (\partial m)g)$. A cat^1-group is, in turn, equivalent to a simplicial group with Moore complex has length 1. The simplicial group is constructed by considering the cat^1-group as a category and taking its nerve. Alternatively, the simplicial group can be constructed by viewing the crossed module as a crossed complex and using a nonabelian version of the Dold-Kan theorem.

The following example concerns the crossed module

$\partial\colon G \to Aut(G), g \mapsto (x \mapsto gxg^{-1})$

associated to the dihedral group G of order 16. This crossed module represents, up to homotopy type, a connected space X with $\pi_i X = 0$ for $i \geq 3$, $\pi_2 X = Z(G)$, $\pi_1 X = Aut(G)/Inn(G)$. The space X can be represented, up to homotopy, by a simplicial group. That simplicial group is used in the example to compute

$H_1(X, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
$H_2(X, \mathbb{Z}) = \mathbb{Z}_2$,
$H_3(X, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
$H_4(X, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
$H_5(X, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

```
———————————————————————— Example ————————————————————————
gap> C:=AutomorphismGroupAsCatOneGroup(DihedralGroup(16));
Cat-1-group with underlying group Group(
[ f1, f2, f3, f4, f5, f6, f7, f8, f9 ] ) .

gap> Size(C);
512
gap> Q:=QuasiIsomorph(C);
Cat-1-group with underlying group Group( [ f9, f8, f1, f2*f3, f5 ] ) .
```

```
gap> Size(Q);
32

gap> N:=NerveOfCatOneGroup(Q,6);
Simplicial group of length 6

gap> K:=ChainComplexOfSimplicialGroup(N);
Chain complex of length 6 in characteristic 0 .

gap> Homology(K,1);
[ 2, 2 ]
gap> Homology(K,2);
[ 2 ]
gap> Homology(K,3);
[ 2, 2, 2 ]
gap> Homology(K,4);
[ 2, 2, 2 ]
gap> Homology(K,5);
[ 2, 2, 2, 2, 2, 2 ]
```

12.2 Eilenberg-MacLane spaces as simplicial groups (not recommended)

The following example concerns the Eilenberg-MacLane space $X = K(\mathbb{Z}_3, 3)$ which is a path-connected space with $\pi_3 X = \mathbb{Z}_3$, $\pi_i X = 0$ for $3 \neq i \geq 1$. This space is represented by a simplicial group, and perturbation techniques are used to compute
$H_7(X, \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

```
─────────────────────────── Example ───────────────────────────
gap> A:=AbelianGroup([3]);;AbelianInvariants(A);
[ 3 ]
gap>  K:=EilenbergMacLaneSimplicialGroup(A,3,8);
Simplicial group of length 8

gap> C:=ChainComplex(K);
Chain complex of length 8 in characteristic 0 .

gap> Homology(C,7);
[ 3, 3 ]
```

12.3 Eilenberg-MacLane spaces as simplicial free abelian groups (recommended)

For integer $n > 1$ and abelian group A the Eilenberg-MacLane space $K(A, n)$ is better represented as a simplicial free abelian group. (The reason is that the functorial bar resolution of A can be replaced in computations by the smaller functorial Chevalley-Eilenberg complex of A, obviating the need for perturbation techniques.)

The following commands compute the integral homology $H_n(K(\mathbb{Z},3),\mathbb{Z})$ for $0 \leq n \leq 16$. (Note that one typically needs fewer than n terms of the Eilenberg-MacLance space to compute its n-th homology – an error is printed if too few terms of the space are available for a given computation.)

```
──────────────────────── Example ────────────────────────
gap> A:=AbelianPcpGroup([0]);; #infinite cyclic group
gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,3,14);
Simplicial free abelian group of length 14

gap> for n in [0..16] do
> Print("Degree ",n," integral homology of K is ",Homology(K,n),"\n");
> od;
Degree 0 integral homology of K is [ 0 ]
Degree 1 integral homology of K is [   ]
Degree 2 integral homology of K is [   ]
Degree 3 integral homology of K is [ 0 ]
Degree 4 integral homology of K is [   ]
Degree 5 integral homology of K is [ 2 ]
Degree 6 integral homology of K is [   ]
Degree 7 integral homology of K is [ 3 ]
Degree 8 integral homology of K is [ 2 ]
Degree 9 integral homology of K is [ 2 ]
Degree 10 integral homology of K is [ 3 ]
Degree 11 integral homology of K is [ 5, 2 ]
Degree 12 integral homology of K is [ 2 ]
Degree 13 integral homology of K is [   ]
Degree 14 integral homology of K is [ 10, 2 ]
Degree 15 integral homology of K is [ 7, 6 ]
Degree 16 integral homology of K is [   ]
```

For an n-connected pointed space X the Freudenthal Suspension Theorem states that the map $X \rightarrow \Omega(\Sigma X)$ induces a map $\pi_k(X) \rightarrow \pi_k(\Omega(\Sigma X))$ which is an isomorphism for $k \leq 2n$ and epimorphism for $k = 2n+1$. Thus the Eilenberg-MacLane space $K(A,n+1)$ can be constructed from the suspension $\Sigma K(A,n)$ by attaching cells in dimensions $\geq 2n+1$. In particular, there is an isomorphism $H_{k-1}(K(A,n),\mathbb{Z}) \rightarrow H_k(K(A,n+1),\mathbb{Z})$ for $k \leq 2n$ and epimorphism for $k = 2n+1$.

For instance, $H_{k-1}(K(\mathbb{Z},3),\mathbb{Z}) \cong H_k(K(\mathbb{Z},4),\mathbb{Z})$ for $k \leq 6$ and $H_6(K(\mathbb{Z},3),\mathbb{Z}) \twoheadrightarrow H_7(K(\mathbb{Z},4),\mathbb{Z})$. This assertion is seen in the following session.

```
──────────────────────── Example ────────────────────────
gap> A:=AbelianPcpGroup([0]);; #infinite cyclic group
gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,4,11);
Simplicial free abelian group of length 11

gap> for n in [0..13] do
> Print("Degree ",n," integral homology of K is ",Homology(K,n),"\n");
> od;
Degree 0 integral homology of K is [ 0 ]
Degree 1 integral homology of K is [   ]
Degree 2 integral homology of K is [   ]
Degree 3 integral homology of K is [   ]
Degree 4 integral homology of K is [ 0 ]
Degree 5 integral homology of K is [   ]
Degree 6 integral homology of K is [ 2 ]
```

```
Degree 7 integral homology of K is [  ]
Degree 8 integral homology of K is [ 3, 0 ]
Degree 9 integral homology of K is [  ]
Degree 10 integral homology of K is [ 2, 2 ]
Degree 11 integral homology of K is [  ]
Degree 12 integral homology of K is [ 5, 12, 0 ]
Degree 13 integral homology of K is [ 2 ]
```

12.4   Elementary theoretical information on $H^*(K(\pi,n),\mathbb{Z})$

The cup product is not implemented for the cohomology ring $H^*(K(\pi,n),\mathbb{Z})$. Standard theoretical spectral sequence arguments have to be applied to obtain basic information relating to the ring structure. To illustrate this the following commands compute $H^n(K(\mathbb{Z},2),\mathbb{Z})$ for the first few values of n.

```
                              Example
  gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,2,10);;
  gap> List([0..10],k->Cohomology(K,k));
  [ [ 0 ], [  ], [ 0 ], [  ], [ 0 ], [  ], [ 0 ], [  ], [ 0 ], [  ], [ 0 ] ]
```

There is a fibration sequence $K(\pi,n) \hookrightarrow * \twoheadrightarrow K(\pi,n+1)$ in which $*$ denotes a contractible space. For $n=1, \pi = \mathbb{Z}$ the terms of the E_2 page of the Serre integral cohomology spectral sequence for this fibration are

- $E_2^{pq} = H^p(K(\mathbb{Z},2),H^q(K(\mathbb{Z},1),\mathbb{Z}))$.

Since $K(\mathbb{Z},1)$ can be taken to be the circle S^1 we know that it has non-trivial cohomology in degrees 0 and 1 only. The first few terms of the E_2 page are given in the following table.

q/p	0	1	2	3	4	5	6	7	8	9	10
1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table: E^2 cohomology page for $K(\mathbb{Z},1) \hookrightarrow * \twoheadrightarrow K(\mathbb{Z},2)$

Let x denote the generator of $H^1(K(\mathbb{Z},1),\mathbb{Z})$ and y denote the generator of $H^2(K(\mathbb{Z},2),\mathbb{Z})$. Since $*$ has zero cohomology in degrees ≥ 1 we see that the differential must restrict to an isomorphism $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ with $d_2(x) = y$. Then we see that the differential must restrict to an isomorphism $d_2 \colon E_2^{2,1} \to E_2^{4,0}$ defined on the generator xy of $E_2^{2,1}$ by

$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)}xd_2(y) = y^2 .$$

Hence $E_2^{4,0} \cong H^4(K(\mathbb{Z},2),\mathbb{Z})$ is generated by y^2. The argument extends to show that $H^6(K(\mathbb{Z},2),\mathbb{Z})$ is generated by y^3, $H^8(K(\mathbb{Z},2),\mathbb{Z})$ is generated by y^4, and so on.

In fact, to obtain a complete description of the ring $H^*(K(\mathbb{Z},2),\mathbb{Z})$ in this fashion there is no benefit to using computer methods at all. We only need to know the cohomology ring $H^*(K(\mathbb{Z},1),\mathbb{Z}) = H^*(S^1,\mathbb{Z})$ and the single cohomology group $H^2(K(\mathbb{Z},2),\mathbb{Z})$.

A similar approach can be attempted for $H^*(K(\mathbb{Z},3),\mathbb{Z})$ using the fibration sequence $K(\mathbb{Z},2) \hookrightarrow * \twoheadrightarrow K(\mathbb{Z},3)$ and, as explained in Chapter 5 of [Hat01], yields the computation of the group

$H^i(K(\mathbb{Z},3),\mathbb{Z})$ for $4 \leq i \leq 13$. The method does not directly yield $H^3(K(\mathbb{Z},3),\mathbb{Z})$ and breaks down in degree 14 yielding only that $H^{14}(K(\mathbb{Z},3),\mathbb{Z}) = 0$ or $\mathbb{Z}_3$. The following commands provide $H^3(K(\mathbb{Z},3),\mathbb{Z}) = \mathbb{Z}$ and $H^{14}(K(\mathbb{Z},3),\mathbb{Z}) = 0$.

```
————————————————————— Example —————————————————————
  gap> A:=AbelianPcpGroup([0]);;
  gap> K:=EilenbergMacLaneSimplicialFreeAbelianGroup(A,3,15);;
  gap> Cohomology(K,3);
  [ 0 ]
  gap> Cohomology(K,14);
  [  ]
```

However, the implementation of these commands is currently a bit naive, and computationally ineffi-cient, since they do not currently employ any homological perturbation techniques.

12.5 The first three non-trivial homotopy groups of spheres

The Hurewicz Theorem immediately gives

$$\pi_n(S^n) \cong \mathbb{Z} \quad (n \geq 1)$$

and

$$\pi_k(S^n) = 0 \quad (k \leq n-1).$$

As a CW-complex the Eilenberg-MacLane space $K = K(\mathbb{Z},n)$ can be obtained from an n-sphere $S^n = e^0 \cup e^n$ by attaching cells in dimensions $\geq n+2$ so as to kill the higher homotopy groups of S^n. From the inclusion $\iota\colon S^n \hookrightarrow K(\mathbb{Z},n)$ we can form the mapping cone $X = C(\iota)$. The long exact homotopy sequence

$$\cdots \to \pi_{k+1}K \to \pi_{k+1}(K,S^n) \to \pi_k S^n \to \pi_k K \to \pi_k(K,S^n) \to \cdots$$

implies that $\pi_k(K,S^n) = 0$ for $0 \leq k \leq n+1$ and $\pi_{n+2}(K,S^n) \cong \pi_{n+1}(S^n)$. The relative Hurewicz Theorem gives an isomorphism $\pi_{n+2}(K,S^n) \cong H_{n+2}(K,S^n,\mathbb{Z})$. The long exact homology sequence

$$\cdots H_{n+2}(S^n,\mathbb{Z}) \to H_{n+2}(K,\mathbb{Z}) \to H_{n+2}(K,S^n,\mathbb{Z}) \to H_{n+1}(S^n,\mathbb{Z}) \to \cdots$$

arising from the cofibration $S^n \hookrightarrow K \twoheadrightarrow X$ implies that $\pi_{n+1}(S^n) \cong \pi_{n+2}(K,S^n) \cong H_{n+2}(K,S^n,\mathbb{Z}) \cong H_{n+2}(K,\mathbb{Z})$. From the GAP computations in 12.3 and the Freudenthal Suspension Theorem we find:

$$\pi_3 S^2 \cong \mathbb{Z}, \qquad \pi_{n+1}(S^n) \cong \mathbb{Z}_2 \quad (n \geq 3).$$

The Hopf fibration $S^3 \to S^2$ has fibre $S^1 = K(\mathbb{Z},1)$. It can be constructed by viewing S^3 as all pairs $(z_1,z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$ and viewing S^2 as $\mathbb{C} \cup \infty$; the map sends $(z_1,z_2) \mapsto z_1/z_2$. The homotopy exact sequence of the Hopf fibration yields $\pi_k(S^3) \cong \pi_k(S^2)$ for $k \geq 3$, and in particular

$$\pi_4(S^2) \cong \pi_4(S^3) \cong \mathbb{Z}_2 \,.$$

It will require further techniques (such as the Postnikov tower argument in Section 12.7 below) to establish that $\pi_5(S^3) \cong \mathbb{Z}_2$. Once we have this isomorphism for $\pi_5(S^3)$, the generalized Hopf fibration $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$ comes into play. This fibration is contructed as for the classical fibration, but using pairs (z_1,z_2) of quaternions rather than pairs of complex numbers. The Hurewicz Theorem gives $\pi_3(S^7) = 0$; the fibre S^3 is thus homotopic to a point in S^7 and the inclusion of the fibre induces the zero homomorphism $\pi_k(S^3) \xrightarrow{0} \pi_k(S^7) \quad (k \geq 1)$. The exact homotopy sequence of the generalized Hopf

fibration then gives $\pi_k(S^4) \cong \pi_k(S^7) \oplus \pi_{k-1}(S^3)$. On taking $k = 6$ we obtain $\pi_6(S^4) \cong \pi_5(S^3) \cong \mathbb{Z}_2$. Freudenthal suspension then gives

$$\pi_{n+2}(S^n) \cong \mathbb{Z}_2, \quad (n \geq 2).$$

12.6 The first two non-trivial homotopy groups of the suspension and double suspension of a $K(G, 1)$

For any group G we consider the homotopy groups $\pi_n(\Sigma K(G, 1))$ of the suspension $\Sigma K(G, 1)$ of the Eilenberg-MacLance space $K(G, 1)$. On taking $G = \mathbb{Z}$, and observing that $S^2 = \Sigma K(\mathbb{Z}, 1)$, we specialize to the homotopy groups of the 2-sphere S^2.

By construction,

$$\pi_1(\Sigma K(G, 1)) = 0 \, .$$

The Hurewicz Theorem gives

$$\pi_2(\Sigma K(G, 1)) \cong G_{ab}$$

via the isomorphisms $\pi_2(\Sigma K(G, 1)) \cong H_2(\Sigma K(G, 1), \mathbb{Z}) \cong H_1(K(G, 1), \mathbb{Z}) \cong G_{ab}$. R. Brown and J.-L. Loday [BL87] obtained the formulae

$$\pi_3(\Sigma K(G, 1)) \cong \ker(G \otimes G \to G, x \otimes y \mapsto [x, y]) \, ,$$

$$\pi_4(\Sigma^2 K(G, 1)) \cong \ker(G \widetilde{\otimes} G \to G, x \widetilde{\otimes} y \mapsto [x, y])$$

involving the nonabelian tensor square and nonabelian symmetric square of the group G. The following commands use the nonabelian tensor and symmetric product to compute the third and fourth homotopy groups for $G = Syl_2(M_{12})$ the Sylow 2-subgroup of the Mathieu group M_{12}.

```
———————————————————————— Example ————————————————————————
gap> G:=SylowSubgroup(MathieuGroup(12),2);;
gap> ThirdHomotopyGroupOfSuspensionB(G);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
gap>
gap> FourthHomotopyGroupOfDoubleSuspensionB(G);
[ 2, 2, 2, 2, 2, 2 ]
```

12.7 Postnikov towers and $\pi_5(S^3)$

A Postnikov system for the sphere S^3 consists of a sequence of fibrations $\cdots X_3 \xrightarrow{p_3} X_2 \xrightarrow{p_2} X_1 \xrightarrow{p_1} *$ and a sequence of maps $\phi_n : S^3 \to X_n$ such that

- $p_n \circ \phi_n = \phi_{n-1}$

- The map $\phi_n : S^3 \to X_n$ induces an isomorphism $\pi_k(S^3) \to \pi_k(X_n)$ for all $k \leq n$

- $\pi_k(X_n) = 0$ for $k > n$

- and consequently each fibration p_n has fibre an Eilenberg-MacLane space $K(\pi_n(S^3), n)$.

The space X_n is obtained from S^3 by adding cells in dimensions $\geq n+2$ and thus

- $H_k(X_n,\mathbb{Z}) = H_k(S^3,\mathbb{Z})$ for $k \leq n+1$.

So in particular $X_1 = X_2 = *, X_3 = K(\mathbb{Z},3)$ and we have a fibration sequence $K(\pi_4(S^3),4) \hookrightarrow X_4 \twoheadrightarrow K(\mathbb{Z},3)$. The terms in the E_2 page of the Serre integral cohomology spectral sequence of this fibration are

- $E_2^{p,q} = H^p(K(\mathbb{Z},3), H_q(K(\mathbb{Z}_2,4),\mathbb{Z}))$.

The first few terms in the E_2 page can be computed using the commands of Sections 12.2 and 12.3 and recorded as follows.

	0	1	2	3	4	5	6	7	8	9
8	$\mathbb{Z}_2$	0	0							
7	$\mathbb{Z}_2$	0	0							
6	0	0	0							
5	$\pi_4(S^3)$	0	0	$\pi_4(S^3)$	0	0	0			
4	0	0	0	0	0	0				
3	0	0	0	0	0	0				
2	0	0	0	0	0	0	0	0		
1	0	0	0	0	0	0	0	0		
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$
q/p	0	1	2	3	4	5	6	7	8	9

Table: E_2 cohomology page for $K(\pi_4(S^3),4) \hookrightarrow X_4 \twoheadrightarrow X_3$

Since we know that $H^5(X_4,\mathbb{Z}) = 0$, the differentials in the spectral sequence must restrict to an isomorphism $E_2^{0,5} = \pi_4(S^3) \xrightarrow{\cong} E_2^{6,0} = \mathbb{Z}_2$. This provides an alternative derivation of $\pi_4(S^3) \cong \mathbb{Z}_2$. We can also immediately deduce that $H^6(X_4,\mathbb{Z}) = 0$. Let x be the generator of $E_2^{0,5}$ and y the generator of $E_2^{3,0}$. Then the generator xy of $E_2^{3,5}$ gets mapped to a non-zero element $d_7(xy) = d_7(x)y - xd_7(y)$. Hence the term $E_2^{0,7} = \mathbb{Z}_2$ must get mapped to zero in $E_2^{3,5}$. It follows that $H^7(X_4,\mathbb{Z}) = \mathbb{Z}_2$.

The integral cohomology of Eilenberg-MacLane spaces yields the following information on the E_2 page $E_2^{p,q} = H_p(X_4, H^q(K(\pi_5 S^3,5),\mathbb{Z}))$ for the fibration $K(\pi_5(S^3),5) \hookrightarrow X_5 \twoheadrightarrow X_4$.

	0	1	2	3	4	5	6	7
6	$\pi_5(S^3)$	0	0	$\pi_5(S^3)$	0	0		
5	0	0	0	0	0	0	0	
4	0	0	0	0	0	0	0	
3	0	0	0	0	0	0	0	
2	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	$H^7(X_4,\mathbb{Z})$
q/p	0	1	2	3	4	5	6	7

Table: E_2 cohomology page for $K(\pi_5(S^3),5) \hookrightarrow X_5 \twoheadrightarrow X_4$

Since we know that $H^6(X_5,\mathbb{Z}) = 0$, the differentials in the spectral sequence must restrict to an isomorphism $E_2^{0,6} = \pi_5(S^3) \xrightarrow{\cong} E_2^{7,0} = H^7(X_4,\mathbb{Z})$. We can conclude the desired result:

$$\pi_5(S^3) = \mathbb{Z}_2 .$$

Note that the fibration $X_4 \twoheadrightarrow K(\mathbb{Z},3)$ is determined by a cohomology class $\kappa \in H^5(K(\mathbb{Z},3),\mathbb{Z}_2) = \mathbb{Z}_2$. If $\kappa = 0$ then we'd have $X_4 = K(\mathbb{Z}_2,4) \times K(\mathbb{Z},3)$ and, as the following commands show, we'd then have $H_4(X_4,\mathbb{Z}) = \mathbb{Z}_2$.

```
─────────────── Example ───────────────
gap> K:=EilenbergMacLaneSimplicialGroup(AbelianPcpGroup([0]),3,7);;
gap> L:=EilenbergMacLaneSimplicialGroup(CyclicGroup(2),4,7);;
gap> CK:=ChainComplex(K);;
gap> CL:=ChainComplex(L);;
gap> T:=TensorProduct(CK,CL);;
gap> Homology(T,4);
[ 2 ]
```

Since we know that $H_4(X_4,\mathbb{Z}) = 0$ we can conclude that the Postnikov invariant κ is the non-zero class in $H^5(K(\mathbb{Z},3),\mathbb{Z}_2) = \mathbb{Z}_2$.

12.8 Towards $\pi_4(\Sigma K(G,1))$

Consider the suspension $X = \Sigma K(G,1)$ of a classifying space of a group G once again. This space has a Postnikov system in which $X_1 = *$, $X_2 = K(G_{ab},2)$. We have a fibration sequence $K(\pi_3 X,3) \hookrightarrow X_3 \twoheadrightarrow K(G_{ab},2)$. The corresponding integral cohomology Serre spectral sequence has E_2 page with terms

- $E_2^{p,q} = H^p(K(G_{ab},2),H^q(K(\pi_3 X,3)),\mathbb{Z}))$.

As an example, for the Alternating group $G = A_4$ of order 12 the following commands of Section 12.6 compute $G_{ab} = \mathbb{Z}_3$ and $\pi_3 X = \mathbb{Z}_6$.

```
─────────────── Example ───────────────
gap> AbelianInvariants(G);
[ 3 ]
gap> ThirdHomotopyGroupOfSuspensionB(G);
[ 2, 3 ]
```

The first terms of the E_2 page can be calculated using the commands of Sections 12.2 and 12.3.

q/p	0	1	2	3	4	5	6	7
7	$\mathbb{Z}_2$	0						
6	$\mathbb{Z}_2$	0	0	0				
5	0	0	0	0				
4	$\mathbb{Z}_6$	0	0	$\mathbb{Z}_3$				
3	0	0	0	0	0	0		
2	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	
0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_9$

Table: E^2 cohomology page for $K(\pi_3 X,3) \hookrightarrow X_3 \twoheadrightarrow X_2$

We know that $H^1(X_3,\mathbb{Z}) = 0$, $H^2(X_3,\mathbb{Z}) = H^1(G,\mathbb{Z}) = 0$, $H^3(X_3,\mathbb{Z}) = H^2(G,\mathbb{Z}) = \mathbb{Z}_3$, and that $H^4(X_3,\mathbb{Z})$ is a subgroup of $H^3(G,\mathbb{Z}) = \mathbb{Z}_2$. It follows that the differential induces a surjection $E_2^{0,4} = \mathbb{Z}_6 \twoheadrightarrow E_2^{5,0} = \mathbb{Z}_3$. Consequently $H^4(X_3,\mathbb{Z}) = \mathbb{Z}_2$ and $H^5(X_3,\mathbb{Z}) = 0$ and $H^6(X_3,\mathbb{Z}) = \mathbb{Z}_2$.

The E_2 page for the fibration $K(\pi_4 X,4) \hookrightarrow X_4 \twoheadrightarrow X_3$ contains the following terms.

5	$\pi_4 X$	0	0				
4	0	0	0	0			
3	0	0	0	0	0	0	
2	0	0	0	0	0	0	
1	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
q/p	0	1	2	3	4	5	6

Table: E^2 cohomology page for $K(\pi_4 X, 4) \hookrightarrow X_4 \twoheadrightarrow X_3$

We know that $H^5(X_4, \mathbb{Z})$ is a subgroup of $H^4(G, \mathbb{Z}) = \mathbb{Z}_6$, and hence that there is a homomorphisms $\pi_4 X \to \mathbb{Z}_2$ whose kernel is a subgroup of $\mathbb{Z}_6$. If follows that $|\pi_4 X| \le 12$.

12.9   Enumerating homotopy 2-types

A *2-type* is a CW-complex X whose homotopy groups are trivial in dimensions $n = 0$ and $n > 2$. As explained in 6.8 the homotopy type of such a space can be captured algebraically by a cat^1-group G. Let X, Y be 2-tytpes represented by cat^1-groups G, H. If X and Y are homotopy equivalent then there exists a sequence of morphisms of cat^1-groups

$$G \to K_1 \to K_2 \leftarrow K_3 \to \cdots \to K_n \leftarrow H$$

in which each morphism induces isomorphisms of homotopy groups. When such a sequence exists we say that G is *quasi-isomorphic* to H. We have the following result.

THEOREM. The 2-types X and Y are homotopy equivalent if and only if the associated cat^1-groups G and H are quasi-isomorphic.

The following commands produce a list L of all of the 62 non-isomorphic cat^1-groups whose underlying group has order 16.

```
──────────── Example ────────────
 gap> L:=[];;
 gap> for G in AllSmallGroups(16) do
 > Append(L,CatOneGroupsByGroup(G));
 > od;
 gap> Length(L);
 62
```

The next commands use the first and second homotopy groups to prove that the list L contains at least 37 distinct quasi-isomorphism types.

```
──────────── Example ────────────
 gap> Invariants:=function(G)
 > local inv;
 > inv:=[];
 > inv[1]:=IdGroup(HomotopyGroup(G,1));
 > inv[2]:=IdGroup(HomotopyGroup(G,2));
 > return inv;
 > end;;

 gap> C:=Classify(L,Invariants);;
 gap> Length(C);
```

The following additional commands use second and third integral homology in conjunction with the first two homotopy groups to prove that the list *L* contains AT LEAST 49 distinct quasi-isomorphism types.

```
――――――――――――――――― Example ―――――――――――――――――
gap> Invariants2:=function(G)
> local inv;
> inv:=[];
> inv[1]:=Homology(G,2);
> inv[2]:=Homology(G,3);
> return inv;
> end;;
gap> C:=RefineClassification(C,Invariants2);;

gap> Length(C);
49
```

The following commands show that the above list *L* contains AT MOST 51 distinct quasi-isomorphism types.

```
――――――――――――――――― Example ―――――――――――――――――
gap> Q:=List(L,QuasiIsomorph);;
gap> M:=[];;

gap> for q in Q do
> bool:=true;;
> for m in M do
> if not IsomorphismCatOneGroups(m,q)=fail then bool:=false; break; fi;
> od;
> if bool then Add(M,q); fi;
> od;

gap> Length(M);
51
```

12.10   Identifying cat^1-groups of low order

Let us define the *order* of a cat^1-group to be the order of its underlying group. The function `IdQuasiCatOneGroup(C)` inputs a cat^1-group *C* of "low order" and returns an integer pair $[n,k]$ that uniquely idenifies the quasi-isomorphism type of *C*. The integer *n* is the order of a smallest cat^1-group quasi-isomorphic to *C*. The integer *k* identifies a particular cat^1-group of order *n*.

The following commands use this function to show that there are precisely 49 distinct quasi-isomorphism types of cat^1-groups of order 16.

```
――――――――――――――――― Example ―――――――――――――――――
gap> L:=[];;
gap> for G in AllSmallGroups(16) do
> Append(L,CatOneGroupsByGroup(G));
> od;
gap> M:=List(L,IdQuasiCatOneGroup);
[ [ 16, 1 ], [ 16, 2 ], [ 16, 3 ], [ 16, 4 ], [ 16, 5 ], [ 4, 4 ], [ 1, 1 ],
```

```
   [ 16, 6 ], [ 16, 7 ], [ 16, 8 ], [ 16, 9 ], [ 16, 10 ], [ 16, 11 ],
   [ 16, 9 ], [ 16, 12 ], [ 16, 13 ], [ 16, 14 ], [ 16, 15 ], [ 4, 1 ],
   [ 4, 2 ], [ 16, 16 ], [ 16, 17 ], [ 16, 18 ], [ 16, 19 ], [ 16, 20 ],
   [ 16, 21 ], [ 16, 22 ], [ 16, 23 ], [ 16, 24 ], [ 16, 25 ], [ 16, 26 ],
   [ 16, 27 ], [ 16, 28 ], [ 4, 3 ], [ 4, 1 ], [ 4, 4 ], [ 4, 4 ], [ 4, 2 ],
   [ 4, 5 ], [ 16, 29 ], [ 16, 30 ], [ 16, 31 ], [ 16, 32 ], [ 16, 33 ],
   [ 16, 34 ], [ 4, 3 ], [ 4, 4 ], [ 4, 4 ], [ 16, 35 ], [ 16, 36 ], [ 4, 3 ],
   [ 16, 37 ], [ 16, 38 ], [ 16, 39 ], [ 16, 40 ], [ 16, 41 ], [ 16, 42 ],
   [ 16, 43 ], [ 4, 3 ], [ 4, 4 ], [ 1, 1 ], [ 4, 5 ] ]
gap> Length(SSortedList(M));
49
```

The next example first identifies the order and the identity number of the cat^1-group C corresponding to the crossed module (see 12.1)

$$\iota: G \longrightarrow Aut(G), g \mapsto (x \mapsto gxg^{-1})$$

for the dihedral group G of order 10. It then realizes a smallest possible cat^1-group D of this quasi-isomorphism type.

```
——————————————————— Example ———————————————————
gap> C:=AutomorphismGroupAsCatOneGroup(DihedralGroup(10));
Cat-1-group with underlying group Group( [ f1, f2, f3, f4, f5 ] ) .

gap> Order(C);
200
gap> IdCatOneGroup(C);
[ 200, 42, 4 ]
gap>
gap> IdQuasiCatOneGroup(C);
[ 2, 1 ]
gap> D:=SmallCatOneGroup(2,1);
Cat-1-group with underlying group Group( [ f1 ] ) .
```

12.11   Identifying crossed modules of low order

The following commands construct the crossed module $\partial: G \otimes G \to G$ involving the nonabelian tensor square of the dihedral group G of order 10, identify it as being number 71 in the list of crossed modules of order 100, create a quasi-isomorphic crossed module of order 4, and finally construct the corresponding cat^1-group of order 100.

```
——————————————————— Example ———————————————————
gap> G:=DihedralGroup(10);;
gap> T:=NonabelianTensorSquareAsCrossedModule(G);
Crossed module with group homomorphism GroupHomomorphismByImages( Group(
[ f3*f1*f3^-1*f1^-1, f3*f2*f3^-1*f2^-1 ] ), Group( [ f1, f2 ] ),
[ f3*f1*f3^-1*f1^-1, f3*f2*f3^-1*f2^-1 ], [ <identity> of ..., f2^3 ] )

gap> IdCrossedModule(T);
[ 100, 71 ]
```

```
gap> Q:=QuasiIsomorph(T);
Crossed module with group homomorphism Pcgs([ f2 ]) -> [ <identity> of ... ]

gap> Order(Q);
4
gap> C:=CatOneGroupByCrossedModule(T);
Cat-1-group with underlying group Group( [ F1, F2, F1 ] ) .
```

Chapter 13

Congruence Subgroups, Cuspidal Cohomology and Hecke Operators

In this chapter we explain how HAP can be used to make computions about modular forms associated to congruence subgroups Γ of $SL_2(\mathbb{Z})$. Also, in Subsection 10.8 onwards, we demonstrate cohomology computations for the *Picard group* $SL_2(\mathbb{Z}[i])$, some *Bianchi groups* $PSL_2(\mathcal{O}_{-d})$ where $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$ for square free positive integer d, and some other groups of the form $SL_m(\mathcal{O})$, $GL_m(\mathcal{O})$, $PSL_m(\mathcal{O})$, $PGL_m(\mathcal{O})$, for $m = 2, 3, 4$ and certain $\mathcal{O} = \mathbb{Z}, \mathcal{O}_{-d}$.

13.1 Eichler-Shimura isomorphism

We begin by recalling the Eichler-Shimura isomorphism [Eic57][Shi59]

$$S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \oplus E_k(\Gamma) \cong_{\mathsf{Hecke}} H^1(\Gamma, P_{\mathbb{C}}(k-2))$$

which relates the cohomology of groups to the theory of modular forms associated to a finite index subgroup Γ of $SL_2(\mathbb{Z})$. In subsequent sections we explain how to compute with the right-hand side of the isomorphism. But first, for completeness, let us define the terms on the left-hand side.

Let N be a positive integer. A subgroup Γ of $SL_2(\mathbb{Z})$ is said to be a *congruence subgroup* of level N if it contains the kernel of the canonical homomorphism $\pi_N : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$. So any congruence subgroup is of finite index in $SL_2(\mathbb{Z})$, but the converse is not true.

One congruence subgroup of particular interest is the group $\Gamma_1(N) = \ker(\pi_N)$, known as the *principal congruence subgroup* of level N. Another congruence subgroup of particular interest is the group $\Gamma_0(N)$ of those matrices that project to upper triangular matrices in $SL_2(\mathbb{Z}/N\mathbb{Z})$.

A *modular form* of weight k for a congruence subgroup Γ is a complex valued function on the upper-half plane, $f : \mathfrak{h} = \{z \in \mathbb{C} : Re(z) > 0\} \to \mathbb{C}$, satisfying:

- $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

- f is 'holomorphic' on the *extended upper-half plane* $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$ obtained from the upper-half plane by 'adjoining a point at each cusp'.

The collection of all weight k modular forms for Γ form a vector space $M_k(\Gamma)$ over $\mathbb{C}$.

A modular form f is said to be a *cusp form* if $f(\infty) = 0$. The collection of all weight k cusp forms for Γ form a vector subspace $S_k(\Gamma)$. There is a decomposition

$$M_k(\Gamma) \cong S_k(\Gamma) \oplus E_k(\Gamma)$$

involving a summand $E_k(\Gamma)$ known as the *Eisenstein space*. See [Ste07] for further introductory details on modular forms.

The Eichler-Shimura isomorphism is more than an isomorphism of vector spaces. It is an isomorphism of Hecke modules: both sides admit notions of *Hecke operators*, and the isomorphism preserves these operators. The bar on the left-hand side of the isomorphism denotes complex conjugation, or *anti-holomorphic* forms. See [Wie78] for a full account of the isomorphism.

On the right-hand side of the isomorphism, the $\mathbb{Z}\Gamma$-module $P_{\mathbb{C}}(k-2) \subset \mathbb{C}[x,y]$ denotes the space of homogeneous degree $k-2$ polynomials with action of Γ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(dx - by, -cx + ay) \ .$$

In particular $P_{\mathbb{C}}(0) = \mathbb{C}$ is the trivial module. Below we shall compute with the integral analogue $P_{\mathbb{Z}}(k-2) \subset \mathbb{Z}[x,y]$.

In the following sections we explain how to use the right-hand side of the Eichler-Shimura isomorphism to compute eigenvalues of the Hecke operators restricted to the subspace $S_k(\Gamma)$ of cusp forms.

13.2 Generators for $SL_2(\mathbb{Z})$ and the cubic tree

The matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$ and it is not difficult to devise an algorithm for expressing an arbitrary integer matrix A of determinant 1 as a word in S, T and their inverses. The following illustrates such an algorithm.

```
────────────────────── Example ──────────────────────
gap> A:=[[4,9],[7,16]];;
gap> word:=AsWordInSL2Z(A);
[ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, 1 ], [ -1, 0 ] ], [ [ 1, -1 ], [ 0, 1 ] ],
  [ [ 0, 1 ], [ -1, 0 ] ], [ [ 1, 1 ], [ 0, 1 ] ], [ [ 0, 1 ], [ -1, 0 ] ],
  [ [ 1, -1 ], [ 0, 1 ] ], [ [ 1, -1 ], [ 0, 1 ] ], [ [ 1, -1 ], [ 0, 1 ] ],
  [ [ 0, 1 ], [ -1, 0 ] ], [ [ 1, 1 ], [ 0, 1 ] ], [ [ 1, 1 ], [ 0, 1 ] ] ]
gap> Product(word);
[ [ 4, 9 ], [ 7, 16 ] ]
```

It is convenient to introduce the matrix $U = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. The matrices S and U also generate $SL_2(\mathbb{Z})$. In fact we have a free presentation $SL_2(\mathbb{Z}) = \langle S, U \mid S^4 = U^6 = 1, S^2 = U^3 \rangle$.

The *cubic tree* $\mathscr{T}$ is a tree (*i.e.* a 1-dimensional contractible regular CW-complex) with countably infinitely many edges in which each vertex has degree 3. We can realize the cubic tree $\mathscr{T}$ by taking the left cosets of $\mathscr{U} = \langle U \rangle$ in $SL_2(\mathbb{Z})$ as vertices, and joining cosets $x\mathscr{U}$ and $y\mathscr{U}$ by an edge if, and only if, $x^{-1}y \in \mathscr{U}S\mathscr{U}$. Thus the vertex $\mathscr{U}$ is joined to $S\mathscr{U}$, $US\mathscr{U}$ and $U^2S\mathscr{U}$. The vertices of this tree are in one-to-one correspondence with all reduced words in S, U and U^2 that, apart from the identity, end in S.

From our realization of the cubic tree $\mathscr{T}$ we see that $SL_2(\mathbb{Z})$ acts on $\mathscr{T}$ in such a way that each vertex is stabilized by a cyclic subgroup conjugate to $\mathscr{U} = \langle U \rangle$ and each edge is stabilized by a cyclic subgroup conjugate to $\mathscr{S} = \langle S \rangle$.

In order to store this action of $SL_2(\mathbb{Z})$ on the cubic tree $\mathscr{T}$ we just need to record the following finite amount of information.

13.3 One-dimensional fundamental domains and generators for congruence subgroups

The modular group $\mathscr{M} = PSL_2(\mathbb{Z})$ is isomorphic, as an abstract group, to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. By the Kurosh subgroup theorem, any finite index subgroup $M \subset \mathscr{M}$ is isomorphic to the free product of finitely many copies of $\mathbb{Z}_2$s, $\mathbb{Z}_3$s and $\mathbb{Z}$s. A subset $\underline{x} \subset M$ is an *independent* set of subgroup generators if M is the free product of the cyclic subgroups $< x >$ as x runs over $\underline{x}$. Let us say that a set of elements in $SL_2(\mathbb{Z})$ is *projectively independent* if it maps injectively onto an independent set of subgroup generators $\underline{x} \subset \mathscr{M}$. The generating set $\{S, U\}$ for $SL_2(\mathbb{Z})$ given in the preceding section is projectively independent.

We are interested in constructing a set of generators for a given congruence subgroup Γ. If a small generating set for Γ is required then we should aim to construct one which is close to being projectively independent.

It is useful to invoke the following general result which follows from a perturbation result about free $\mathbb{Z}G$-resolutons in [EHS06, Theorem 2] and an old observation of John Milnor that a free $\mathbb{Z}G$-resolution can be realized as the cellular chain complex of a CW-complex if it can be so realized in low dimensions.

THEOREM. Let X be a contractible CW-complex on which a group G acts by permuting cells. The cellular chain complex C_*X is a $\mathbb{Z}G$-resolution of $\mathbb{Z}$ which typically is not free. Let $[e^n]$ denote the orbit of the n-cell e^n under the action. Let $G^{e^n} \leq G$ denote the stabilizer subgroup of e^n, in which group elements are not required to stabilize e^n point-wise. Let Y_{e^n} denote a contractible CW-complex on which G^{e^n} acts cellularly and freely. Then there exists a contractible CW-complex W on which G acts cellularly and freely, and in which the orbits of n-cells are labelled by $[e^p] \otimes [f^q]$ where $p + q = n$ and $[e^p]$ ranges over the G-orbits of p-cells in X, $[f^q]$ ranges over the G^{e^p}-orbits of q-cells in Y_{e^p}.

Let W be as in the theorem. Then the quotient CW-complex $B_G = W/G$ is a classifying space for G. Let T denote a maximal tree in the 1-skeleton B_G^1. Basic geometric group theory tells us that the 1-cells in $B_G^1 \setminus T$ correspond to a generating set for G.

Suppose we wish to compute a set of generators for a principal congruence subgroup $\Gamma = \Gamma_1(N)$. In the above theorem take $X = \mathscr{T}$ to be the cubic tree, and note that Γ acts freely on $\mathscr{T}$ and thus that $W = \mathscr{T}$. To determine the 1-cells of $B_\Gamma \setminus T$ we need to determine a cellular subspace $D_\Gamma \subset \mathscr{T}$ whose images under the action of Γ cover $\mathscr{T}$ and are pairwise either disjoint or identical. The subspace D_Γ will not be a CW-complex as it won't be closed, but it can be chosen to be connected, and hence contractible. We call D_Γ a *fundamental region* for Γ. We denote by $\mathring{D}_\Gamma$ the largest CW-subcomplex of D_Γ. The vertices of $\mathring{D}_\Gamma$ are the same as the vertices of D_Γ. Thus $\mathring{D}_\Gamma$ is a subtree of the cubic tree with $|\Gamma|/6$ vertices. For each vertex v in the tree $\mathring{D}_\Gamma$ define $\eta(v) = 3 - \text{degree}(v)$. Then the number of generators for Γ will be $(1/2) \sum_{v \in \mathring{D}_\Gamma} \eta(v)$.

The following commands determine projectively independent generators for $\Gamma_1(6)$ and display $\mathring{D}_{\Gamma_1(6)}$. The subgroup $\Gamma_1(6)$ is free on 13 generators.

```
───────────────────────── Example ─────────────────────────
  gap> G:=HAP_PrincipalCongruenceSubgroup(6);;
```

```
gap> HAP_SL2TreeDisplay(G);


gap> gens:=GeneratorsOfGroup(G);
[ [ [ -83, -18 ], [ 60, 13 ] ], [ [ -77, -18 ], [ 30, 7 ] ],
  [ [ -65, -12 ], [ 168, 31 ] ], [ [ -53, -12 ], [ 84, 19 ] ],
  [ [ -47, -18 ], [ 222, 85 ] ], [ [ -41, -12 ], [ 24, 7 ] ],
  [ [ -35, -6 ], [ 6, 1 ] ], [ [ -11, -18 ], [ 30, 49 ] ],
  [ [ -11, -6 ], [ 24, 13 ] ], [ [ -5, -18 ], [ 12, 43 ] ],
  [ [ -5, -12 ], [ 18, 43 ] ], [ [ -5, -6 ], [ 6, 7 ] ],
  [ [ 1, 0 ], [ -6, 1 ] ] ]
```

An alternative but very related approach to computing generators of congruence subgroups of $SL_2(\mathbb{Z})$ is described in [Kul91].

The congruence subgroup $\Gamma_0(N)$ does not act freely on the vertices of $\mathcal{T}$, and so one needs to incorporate a generator for the cyclic stabilizer group according to the above theorem. Alternatively, we can replace the cubic tree by a six-fold cover $\mathcal{T}'$ on whose vertex set $\Gamma_0(N)$ acts freely. This alternative approach will produce a redundant set of generators. The following commands display $\mathring{D}_{\Gamma_0(39)}$ for a fundamental region in $\mathcal{T}'$. They also use the corresponding generating set for $\Gamma_0(39)$, involving 18 generators, to compute the abelianization $\Gamma_0(39)^{ab} = \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}^9$. The abelianization shows that any generating set has at least 11 generators.

```
─────────────────────── Example ───────────────────────
gap> G:=HAP_CongruenceSubgroupGamma0(39);;
gap> HAP_SL2TreeDisplay(G);
gap> Length(GeneratorsOfGroup(G));
18
gap> AbelianInvariants(G);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 3, 3 ]
```

Note that to compute D_Γ one only needs to be able to test whether a given matrix lies in Γ or not. Given an inclusion $\Gamma' \subset \Gamma$ of congruence subgroups, it is straightforward to use the trees $\mathring{D}_{\Gamma'}$ and $\mathring{D}_\Gamma$ to compute a system of coset representative for $\Gamma' \setminus \Gamma$.

13.4 Cohomology of congruence subgroups

To compute the cohomology $H^n(\Gamma, A)$ of a congruence subgroup Γ with coefficients in a $\mathbb{Z}\Gamma$-module A we need to construct $n+1$ terms of a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$. We can do this by first using perturbation techniques (as described in [BE14]) to combine the cubic tree with resolutions for the cyclic groups of order 4 and 6 in order to produce a free $\mathbb{Z}G$-resolution R_* for $G = SL_2(\mathbb{Z})$. This resolution is also a free $\mathbb{Z}\Gamma$-resolution with each term of rank

$$\text{rank}_{\mathbb{Z}\Gamma} R_k = |G : \Gamma| \times \text{rank}_{\mathbb{Z}G} R_k .$$

For congruence subgroups of lowish index in G this resolution suffices to make computations. The following commands compute

$$H^1(\Gamma_0(39), \mathbb{Z}) = \mathbb{Z}^9 .$$

```
                              ─── Example ───
gap> R:=ResolutionSL2Z_alt(2);
Resolution of length 2 in characteristic 0 for SL(2,Integers) .

gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
gap> S:=ResolutionFiniteSubgroup(R,gamma);
Resolution of length 2 in characteristic 0 for
CongruenceSubgroupGamma0( 39)   .

gap> Cohomology(HomToIntegers(S),1);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

This computation establishes that the space $M_2(\Gamma_0(39))$ of weight 2 modular forms is of dimension 9.

The following commands show that $\operatorname{rank}_{\mathbb{Z}\Gamma_0(39)} R_1 = 112$ but that it is possible to apply 'Tietze like' simplifications to R_* to obtain a free $\mathbb{Z}\Gamma_0(39)$-resolution T_* with $\operatorname{rank}_{\mathbb{Z}\Gamma_0(39)} T_1 = 11$. It is more efficient to work with T_* when making cohomology computations with coefficients in a module A of large rank.

```
                              ─── Example ───
gap> S!.dimension(1);
112
gap> T:=TietzeReducedResolution(S);
Resolution of length 2 in characteristic 0 for CongruenceSubgroupGamma0(
39)   .

gap> T!.dimension(1);
11
```

The following commands compute

$$H^1(\Gamma_0(39), P_{\mathbb{Z}}(8)) = \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{168} \oplus \mathbb{Z}^{84} ,$$

$$H^1(\Gamma_0(39), P_{\mathbb{Z}}(9)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

```
                              ─── Example ───
gap> P:=HomogeneousPolynomials(gamma,8);;
gap> c:=Cohomology(HomToIntegralModule(T,P),1);
[ 3, 6, 168, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Length(c);
87

gap> P:=HomogeneousPolynomials(gamma,9);;
gap> c:=Cohomology(HomToIntegralModule(T,P),1);
[ 2, 2 ]
```

This computation establishes that the space $M_{10}(\Gamma_0(39))$ of weight 10 modular forms is of dimension 84, and $M_{11}(\Gamma_0(39))$ is of dimension 0. (There are never any modular forms of odd weight, and so $M_k(\Gamma) = 0$ for all odd k and any congruence subgroup Γ.)

13.4.1 Cohomology with rational coefficients

To calculate cohomology $H^n(\Gamma, A)$ with coefficients in a $\mathbb{Q}\Gamma$-module A it suffices to construct a resolution of $\mathbb{Z}$ by non-free $\mathbb{Z}\Gamma$-modules where Γ acts with finite stabilizer groups on each module in the resolution. Computing over $\mathbb{Q}$ is computationally less expensive than computing over $\mathbb{Z}$. The following commands first compute $H^1(\Gamma_0(39), \mathbb{Q}) = H_1(\Gamma_0(39), \mathbb{Q}) = \mathbb{Q}^9$. As a larger example, they then compute $H^1(\Gamma_0(2^{13}-1), \mathbb{Q}) = \mathbb{Q}^{1365}$ where $\Gamma_0(2^{13}-1)$ has index 8192 in $SL_2(\mathbb{Z})$.

```
------------------------- Example -------------------------
 gap> K:=ContractibleGcomplex("SL(2,Z)");
 Non-free resolution in characteristic 0 for SL(2,Integers) .

 gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
 gap> KK:=NonFreeResolutionFiniteSubgroup(K,gamma);
 Non-free resolution in characteristic 0 for <matrix group with
 18 generators> .

 gap> C:=TensorWithRationals(KK);
 gap> Homology(C,1);
 9

 gap> G:=HAP_CongruenceSubgroupGamma0(2^13-1);;
 gap> IndexInSL2Z(G);
 8192
 gap> KK:=NonFreeResolutionFiniteSubgroup(K,G);;
 gap> C:=TensorWithRationals(KK);;
 gap> Homology(C,1);
 1365
```

13.5 Cuspidal cohomology

To define and compute cuspidal cohomology we consider the action of $SL_2(\mathbb{Z})$ on the upper-half plane $\mathfrak{h}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} .$$

A standard 'fundamental domain' for this action is the region

$$\begin{aligned} D = \quad & \{z \in \mathfrak{h} \; : \; |z| > 1, |\text{Re}(z)| < \tfrac{1}{2}\} \\ & \cup \{z \in \mathfrak{h} \; : \; |z| \geq 1, \text{Re}(z) = -\tfrac{1}{2}\} \\ & \cup \{z \in \mathfrak{h} \; : \; |z| = 1, -\tfrac{1}{2} \leq \text{Re}(z) \leq 0\} \end{aligned}$$

illustrated below.

The action factors through an action of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$. The images of D under the action of $PSL_2(\mathbb{Z})$ cover the upper-half plane, and any two images have at most a single point

in common. The possible common points are the bottom left-hand corner point which is stabilized by $\langle U \rangle$, and the bottom middle point which is stabilized by $\langle S \rangle$.

A congruence subgroup Γ has a 'fundamental domain' D_Γ equal to a union of finitely many copies of D, one copy for each coset in $\Gamma \backslash SL_2(\mathbb{Z})$. The quotient space $X = \Gamma \backslash \mathfrak{h}$ is not compact, and can be compactified in several ways. We are interested in the Borel-Serre compactification. This is a space X^{BS} for which there is an inclusion $X \hookrightarrow X^{BS}$ and this inclusion is a homotopy equivalence. One defines the *boundary* $\partial X^{BS} = X^{BS} - X$ and uses the inclusion $\partial X^{BS} \hookrightarrow X^{BS} \simeq X$ to define the cuspidal cohomology group, over the ground ring $\mathbb{C}$, as

$$H^n_{cusp}(\Gamma, P_\mathbb{C}(k-2)) = \ker(\ H^n(X, P_\mathbb{C}(k-2)) \to H^n(\partial X^{BS}, P_\mathbb{C}(k-2))\ ).$$

Strictly speaking, this is the definition of *interior cohomology* $H^n_!(\Gamma, P_\mathbb{C}(k-2))$ which in general contains the cuspidal cohomology as a subgroup. However, for congruence subgroups of $SL_2(\mathbb{Z})$ there is equality $H^n_!(\Gamma, P_\mathbb{C}(k-2)) = H^n_{cusp}(\Gamma, P_\mathbb{C}(k-2))$.

Working over $\mathbb{C}$ has the advantage of avoiding the technical issue that Γ does not necessarily act freely on $\mathfrak{h}$ since there are points with finite cyclic stabilizer groups in $SL_2(\mathbb{Z})$. But it has the disadvantage of losing information about torsion in cohomology. So HAP confronts the issue by working with a contractible CW-complex $\tilde{X}^{BS}$ on which Γ acts freely, and Γ-equivariant inclusion $\partial \tilde{X}^{BS} \hookrightarrow \tilde{X}^{BS}$. The definition of cuspidal cohomology that we use, which coincides with the above definition when working over $\mathbb{C}$, is

$$H^n_{cusp}(\Gamma, A) = \ker(\ H^n(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*(\tilde{X}^{BS}), A)) \to H^n(\ \mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*(\tilde{\partial}X^{BS}), A)\ )\ ).$$

The following data is recorded and, using perturbation theory, is combined with free resolutions for C_4 and C_6 to constuct $\tilde{X}^{BS}$.

The following commands calculate

$$H^1_{cusp}(\Gamma_0(39), \mathbb{Z}) = \mathbb{Z}^6 \ .$$

```
——————————— Example ———————————
gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
gap> k:=2;; deg:=1;; c:=CuspidalCohomologyHomomorphism(gamma,deg,k);
[ g1, g2, g3, g4, g5, g6, g7, g8, g9 ] -> [ g1^-1*g3, g1^-1*g3, g1^-1*g3,
  g1^-1*g3, g1^-1*g2, g1^-1*g3, g1^-1*g4, g1^-1*g4, g1^-1*g4 ]
gap> AbelianInvariants(Kernel(c));
[ 0, 0, 0, 0, 0, 0 ]
```

From the Eichler-Shimura isomorphism and the already calculated dimension of $M_2(\Gamma_0(39)) \cong \mathbb{C}^9$, we deduce from this cuspidal cohomology that the space $S_2(\Gamma_0(39))$ of cuspidal weight 2 forms is of dimension 3, and the Eisenstein space $E_2(\Gamma_0(39)) \cong \mathbb{C}^3$ is of dimension 3.

The following commands show that the space $S_4(\Gamma_0(39))$ of cuspidal weight 4 forms is of dimension 12.

```
——————————— Example ———————————
gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
gap> k:=4;; deg:=1;; c:=CuspidalCohomologyHomomorphism(gamma,deg,k);;
gap> AbelianInvariants(Kernel(c));
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

13.6 Hecke operators on forms of weight 2

A congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$ and element $g \in SL_2(\mathbb{Q})$ determine the subgroup $\Gamma' = \Gamma \cap g\Gamma g^{-1}$ and homomorphisms

$$\Gamma \hookleftarrow \Gamma' \xrightarrow{\gamma \mapsto g^{-1}\gamma g} g^{-1}\Gamma'g \hookrightarrow \Gamma .$$

These homomorphisms give rise to homomorphisms of cohomology groups

$$H^n(\Gamma,\mathbb{Z}) \xleftarrow{tr} H^n(\Gamma',\mathbb{Z}) \xleftarrow{\alpha} H^n(g^{-1}\Gamma'g,\mathbb{Z}) \xleftarrow{\beta} H^n(\Gamma,\mathbb{Z})$$

with α, β functorial maps, and tr the transfer map. We define the composite $T_g = tr \circ \alpha \circ \beta \colon H^n(\Gamma,\mathbb{Z}) \to H^n(\Gamma,\mathbb{Z})$ to be the *Hecke operator* determined by g. Further details on this description of Hecke operators can be found in [Ste07, Appendix by P. Gunnells].

For each prime integer $p \geq 1$ we set $T_p = T_g$ with for $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

The following commands compute T_2 and T_5 for $n = 1$ and $\Gamma = \Gamma_0(39)$. The commands also compute the eigenvalues of these two Hecke operators. The final command confirms that T_2 and T_5 commute. (It is a fact that $T_p T_q = T_q T_p$ for all primes p, q.)

```
                          ───── Example ─────
  gap> gamma:=HAP_CongruenceSubgroupGamma0(39);;
  gap> p:=2;;N:=1;;h:=HeckeOperatorWeight2(gamma,p,N);;
  gap> AbelianInvariants(Source(h));
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
  gap> T2:=HomomorphismAsMatrix(h);;
  gap> Display(T2);
  [ [  -2,  -2,   2,   2,   1,   2,   0,   0,   0 ],
    [  -2,   0,   1,   2,  -2,   2,   2,   2,  -2 ],
    [  -2,  -1,   2,   2,  -1,   2,   1,   1,  -1 ],
    [  -2,  -1,   2,   2,   1,   1,   0,   0,   0 ],
    [  -1,   0,   0,   2,  -3,   2,   3,   3,  -3 ],
    [   0,   1,   1,   1,  -1,   0,   1,   1,  -1 ],
    [  -1,   1,   1,  -1,   0,   1,   2,  -1,   1 ],
    [  -1,  -1,   0,   2,  -3,   2,   1,   4,  -1 ],
    [   0,   1,   0,  -1,  -2,   1,   1,   1,   2 ] ]
  gap> Eigenvalues(Rationals,T2);
  [ 3, 1 ]

  gap> p:=5;;N:=1;;h:=HeckeOperator(gamma,p,N);;
  gap> T5:=HomomorphismAsMatrix(h);;
  gap> Display(T5);
  [ [  -1,  -1,   3,   4,   0,   0,   1,   1,  -1 ],
    [  -5,  -1,   5,   4,   0,   0,   3,   3,  -3 ],
    [  -2,   0,   4,   4,   1,   0,  -1,  -1,   1 ],
    [  -2,   0,   3,   2,  -3,   2,   4,   4,  -4 ],
    [  -4,  -2,   4,   4,   3,   0,   1,   1,  -1 ],
    [  -6,  -4,   5,   6,   1,   2,   2,   2,  -2 ],
    [   1,   5,   0,  -4,  -3,   2,   5,  -1,   1 ],
    [  -2,  -2,   2,   4,   0,   0,  -2,   4,   2 ],
    [   1,   3,   0,  -4,  -4,   2,   2,   2,   4 ] ]
  gap> Eigenvalues(Rationals,T5);
  [ 6, 2 ]
```

```
gap>T2*T5=T5*T2;
true
```

13.7   Hecke operators on forms of weight ≥ 2

The above definition of Hecke operator T_g for $g \in SL_2(\mathbb{Q})$ extends to a Hecke operator $T_g: H^1(\Gamma, P_{\mathbb{Q}}(k - 2)) \to H^1(\Gamma, P_{\mathbb{Q}}(k-2))$ for $k \geq 2$. We work over the rationals rather than the integers in order to ensure that the action of Γ on homogeneous polynomials extends to an action of g. The following commands compute the matrix of $T_2: H^1(\Gamma, P_{\mathbb{Q}}(k-2)) \to H^1(\Gamma, P_{\mathbb{Q}}(k-2))$ for $\Gamma = SL_2(\mathbb{Z})$ and $k = 4$;

```
───────────────────────── Example ─────────────────────────
 gap> H:=HAP_CongruenceSubgroupGamma0(1);;
 gap> h:=HeckeOperator(H,2,12);;Display(h);
 [ [   2049,  -7560,      0 ],
   [      0,    -24,      0 ],
   [      0,      0,    -24 ] ]
```

13.8   Reconstructing modular forms from cohomology computations

Given a modular form $f: \mathfrak{h} \to \mathbb{C}$ associated to a congruence subgroup Γ, and given a compact edge e in the tessellation of $\mathfrak{h}$ (*i.e.* an edge in the cubic tree $\mathcal{T}$) arising from the above fundamental domain for $SL_2(\mathbb{Z})$, we can evaluate

$$\int_e f(z)\, dz \; .$$

In this way we obtain a cochain $f_1: C_1(\mathcal{T}) \to \mathbb{C}$ in $Hom_{\mathbb{Z}\Gamma}(C_1(\mathcal{T}), \mathbb{C})$ representing a cohomology class $c(f) \in H^1(Hom_{\mathbb{Z}\Gamma}(C_*(\mathcal{T}), \mathbb{C})) = H^1(\Gamma, \mathbb{C})$. The correspondence $f \mapsto c(f)$ underlies the Eichler-Shimura isomorphism. Hecke operators can be used to recover modular forms from cohomology classes.

Hecke operators restrict to operators on cuspidal cohomology. On the left-hand side of the Eichler-Shimura isomorphism Hecke operators restrict to operators $T_s: S_2(\Gamma) \to S_2(\Gamma)$ for $s \geq 1$.

Let us now introduce the function $q = q(z) = e^{2\pi i z}$ which is holomorphic on $\mathbb{C}$. For any modular form $f(z)$ there are numbers a_s such that

$$f(z) = \sum_{s=0}^{\infty} a_s q^s$$

for all $z \in \mathfrak{h}$. The form f is a cusp form if $a_0 = 0$.

A non-zero cusp form $f \in S_2(\Gamma)$ is an *eigenform* if it is simultaneously an eigenvector for the Hecke operators T_s for all $s = 1, 2, 3, \cdots$. An eigenform is said to be *normalized* if its coefficient $a_1 = 1$. It turns out that if f is a normalized eigenform then the coefficient a_s is an eigenvalue for T_s (see for instance [Ste07] for details). It can be shown [AL70] that $f \in S_2(\Gamma_0(N))$ admits a basis of eigenforms.

This all implies that, in principle, we can construct an approximation to an explicit basis for the space $S_2(\Gamma)$ of cusp forms by computing eigenvalues for Hecke operators.

Suppose that we would like a basis for $S_2(\Gamma_0(11))$. The following commands first show that $H^1_{cusp}(\Gamma_0(11),\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ from which we deduce that $S_2(\Gamma_0(11)) = \mathbb{C}$ is 1-dimensional. Then eigenvalues of Hecke operators are calculated to establish that the modular form

$$f = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + -2q^9 - 2q^{10} + \cdots$$

constitutes a basis for $S_2(\Gamma_0(11))$.

```
———————————————————— Example ————————————————————
gap> gamma:=HAP_CongruenceSubgroupGamma0(11);;
gap> AbelianInvariants(Kernel(CuspidalCohomologyHomomorphism(gamma,1,2)));
[ 0, 0 ]

gap> T1:=HomomorphismAsMatrix(HeckeOperator(gamma,1,1));; Display(T1);
[ [  1,  0,  0 ],
  [  0,  1,  0 ],
  [  0,  0,  1 ] ]
gap> T2:=HomomorphismAsMatrix(HeckeOperator(gamma,2,1));; Display(T2);
[ [  3, -4,  4 ],
  [  0, -2,  0 ],
  [  0,  0, -2 ] ]
gap> T3:=HomomorphismAsMatrix(HeckeOperator(gamma,3,1));; Display(T3);
[ [  4, -4,  4 ],
  [  0, -1,  0 ],
  [  0,  0, -1 ] ]
gap> T5:=HomomorphismAsMatrix(HeckeOperator(gamma,5,1));; Display(T5);
[ [  6, -4,  4 ],
  [  0,  1,  0 ],
  [  0,  0,  1 ] ]
gap> T7:=HomomorphismAsMatrix(HeckeOperator(gamma,7,1));; Display(T7);
[ [  8, -8,  8 ],
  [  0, -2,  0 ],
  [  0,  0, -2 ] ]
```

For a normalized eigenform $f = 1 + \sum_{s=2}^{\infty} a_s q^s$ the coefficients a_s with s a composite integer can be expressed in terms of the coefficients a_p for prime p. If r,s are coprime then $T_{rs} = T_r T_s$. If p is a prime that is not a divisor of the level N of Γ then $a_{p^m} = a_{p^{m-1}} a_p - p a_{p^{m-2}}$. If the prime p divides N then $a_{p^m} = (a_p)^m$. It thus suffices to compute the coefficients a_p for prime integers p only.

The following commands establish that $S_{12}(SL_2(\mathbb{Z}))$ has a basis consisting of one cusp eigenform
$q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9$
$-115920q^{10} + 534612q^{11} - 370944q^{12} - 577738q^{13} + 401856q^{14} + 1217160q^{15} + 987136q^{16}$
$-6905934q^{17} + 2727432q^{18} + 10661420q^{19} + ...$

```
———————————————————— Example ————————————————————
gap> H:=HAP_CongruenceSubgroupGamma0(1);;
gap> for p in [2,3,5,7,11,13,17,19] do
> T:=HeckeOperator(H,p,1,12);;
> Print("eigenvalues= ",Eigenvalues(Rationals,T), " and eigenvectors = ", Eigenvectors(Rationals,
> od;
eigenvalues= [ 2049, -24 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] f
eigenvalues= [ 177148, 252 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ]
eigenvalues= [ 48828126, 4830 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ], [ 0, 0, 1
```

```
eigenvalues= [ 1977326744, -16744 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ], [ 0, 0
eigenvalues= [ 285311670612, 534612 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ], [ 0,
eigenvalues= [ 1792160394038, -577738 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ], [
eigenvalues= [ 34271896307634, -6905934 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ],
eigenvalues= [ 116490258898220, 10661420 ] and eigenvectors = [ [ 1, -2520/691, 0 ], [ 0, 1, 0 ],
```

13.9   The Picard group

Let us now consider the *Picard group* $G = SL_2(\mathbb{Z}[i])$ and its action on *upper-half space*

$$\mathfrak{h}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\} .$$

To describe the action we introduce the symbol j satisfying $j^2 = -1$, $ij = -ji$ and write $z+tj$ instead of (z,t). The action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z+tj) = (a(z+tj)+b)(c(z+tj)+d)^{-1} .$$

Alternatively, and more explicitly, the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z+tj) = \frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2}{|cz+d|^2 + |c|^2 y^2} + \frac{y}{|cz+d|^2 + |c|^2 y^2} j .$$

A standard 'fundamental domain' D for this action is the following region (with some of the boundary points removed).

$$\{z+tj \in \mathfrak{h}^3 \mid 0 \le |\mathrm{Re}(z)| \le \frac{1}{2}, 0 \le \mathrm{Im}(z) \le \frac{1}{2}, z\bar{z}+t^2 \ge 1\}$$

The four bottom vertices of D are $a = -\frac{1}{2} + \frac{1}{2}i + \frac{\sqrt{2}}{2}j$, $b = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$, $c = \frac{1}{2} + \frac{\sqrt{3}}{2}j$, $d = \frac{1}{2} + \frac{1}{2}i + \frac{\sqrt{2}}{2}j$.

The upper-half space $\mathfrak{h}^3$ can be retracted onto a 2-dimensional subspace $\mathscr{T} \subset \mathfrak{h}^3$. The space $\mathscr{T}$ is a contractible 2-dimensional regular CW-complex, and the action of the Picard group G restricts to a cellular action of G on $\mathscr{T}$. Under this action there is one orbit of 2-cells, represented by the curvilinear square with vertices a, b, c and d in the picture. This 2-cell has stabilizer group isomorphic to the quaternion group Q_4 of order 8. There are two orbits of 1-cells, both with stabilizer group isomorphic to a semi-direct product $C_3 : C_4$. There is one orbit of 0-cells, with stabilizer group isomorphic to $SL(2,3)$.

Using perturbation techniques, the 2-complex $\mathscr{T}$ can be combined with free resolutions for the cell stabilizer groups to contruct a regular CW-complex X on which the Picard group G acts freely. The following commands compute the first few terms of the free $\mathbb{Z}G$-resolution $R_* = C_*X$. Then R_* is used to compute

$$H^1(G,\mathbb{Z}) = 0 ,$$

$$H^2(G,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 ,$$

$$H^3(G,\mathbb{Z}) = \mathbb{Z}_6 ,$$

$$H^4(G, \mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_{24} \,,$$

and compute a free presentation for G involving four generators and seven relators.

```
─────────────────── Example ───────────────────
gap> K:=ContractibleGcomplex("SL(2,O-1)");;
gap> R:=FreeGResolution(K,5);;
gap> Cohomology(HomToIntegers(R),1);
[  ]
gap> Cohomology(HomToIntegers(R),2);
[ 2, 2 ]
gap> Cohomology(HomToIntegers(R),3);
[ 6 ]
gap> Cohomology(HomToIntegers(R),4);
[ 4, 24 ]
gap> P:=PresentationOfResolution(R);
rec( freeGroup := <free group on the generators [ f1, f2, f3, f4 ]>,
  gens := [ 184, 185, 186, 187 ],
  relators := [ f1^2*f2^-1*f1^-1*f2^-1, f1*f2*f1*f2^-2,
      f3*f2^2*f1*(f2*f1^-1)^2*f3^-1*f1^2*f2^-2,
      f1*(f2*f1^-1)^2*f3^-1*f1^2*f2^-1*f3^-1,
      f4*f2*f1*(f2*f1^-1)^2*f4^-1*f1*f2^-1, f1*f4^-1*f1^-2*f4^-1,
      f3*f2*f1*(f2*f1^-1)^2*f4^-1*f1*f2^-1*f3^-1*f4*f2 ] )
```

We can also compute the cohomology of $G = SL_2(\mathbb{Z}[i])$ with coefficients in a module such as the module $P_{\mathbb{Z}[i]}(k)$ of degree k homogeneous polynomials with coefficients in $\mathbb{Z}[i]$ and with the action described above. For instance, the following commands compute

$$H^1(G, P_{\mathbb{Z}[i]}(24)) = (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{40} \oplus \mathbb{Z}_{80} \,,$$

$$H^2(G, P_{\mathbb{Z}[i]}(24)) = (\mathbb{Z}_2)^{24} \oplus \mathbb{Z}_{520030} \oplus \mathbb{Z}_{1040060} \oplus \mathbb{Z}^2 \,,$$

$$H^3(G, P_{\mathbb{Z}[i]}(24)) = (\mathbb{Z}_2)^{22} \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_{12})^2 \,.$$

```
─────────────────── Example ───────────────────
gap> G:=R!.group;;
gap> M:=HomogeneousPolynomials(G,24);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,1);
[ 2, 2, 2, 2, 4, 8, 40, 80 ]
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  520030, 1040060, 0, 0 ]
gap> Cohomology(C,3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 12, 12
 ]
```

13.10 Bianchi groups

The *Bianchi groups* are the groups $G = PSL_2(\mathscr{O}_{-d})$ where d is a square free positive integer and $\mathscr{O}_{-d}$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. More explicitly,

$$\mathscr{O}_{-d} = \mathbb{Z}\left[\sqrt{-d}\right] \qquad \text{if } d \equiv 1 \bmod 4,$$

$$\mathscr{O}_{-d} = \mathbb{Z}\left[\frac{1+\sqrt{-d}}{2}\right] \qquad \text{if } d \equiv 2,3 \bmod 4.$$

These groups act on upper-half space $\mathfrak{h}^3$ in the same way as the Picard group. Upper-half space can be tessellated by a 'fundamental domain' for this action. Moreover, as with the Picard group, this tessellation contains a 2-dimensional cellular subspace $\mathscr{T} \subset \mathfrak{h}^3$ where $\mathscr{T}$ is a contractible CW-complex on which G acts cellularly. It should be mentioned that the fundamental domain and the contractible 2-complex $\mathscr{T}$ are not uniquely determined by G. Various algorithms exist for computing $\mathscr{T}$ and its cell stabilizers. One algorithm due to Swan [Swa71] has been implemented by Alexander Rahm [Rah10] and the output for various values of d are stored in HAP. Another approach is to use Voronoi's theory of perfect forms. This approach has been implemented by Sebastian Schoennenbeck [BCNS15] and, again, its output for various values of d are stored in HAP. The following commands combine data from Schoennenbeck's algorithm with free resolutions for cell stabiliers to compute

$$H^1(PSL_2(\mathscr{O}_{-6}), P_{\mathscr{O}_{-6}}(24)) = (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{9240} \oplus \mathbb{Z}_{55440} \oplus \mathbb{Z}^4,$$

$$H^2(PSL_2(\mathscr{O}_{-6}), P_{\mathscr{O}_{-6}}(24)) = \begin{array}{l} (\mathbb{Z}_2)^{26} \oplus (Z_6)^8 \oplus (Z_{12})^9 \oplus \mathbb{Z}_{24} \oplus (\mathbb{Z}_{120})^2 \oplus (\mathbb{Z}_{840})^3 \\ \oplus \mathbb{Z}_{2520} \oplus (\mathbb{Z}_{27720})^2 \oplus (\mathbb{Z}_{24227280})^2 \oplus (\mathbb{Z}_{411863760})^2 \\ \oplus \mathbb{Z}_{245443824374892865187742514283666449812984 0} \\ \oplus \mathbb{Z}_{14726629462493571911264550857019986988779040} \\ \oplus \mathbb{Z}^4 \end{array},$$

$$H^3(PSL_2(\mathscr{O}_{-6}), P_{\mathscr{O}_{-6}}(24)) = (\mathbb{Z}_2)^{23} \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_{12})^2.$$

Note that the action of $SL_2(\mathscr{O}_{-d})$ on $P_{\mathscr{O}_{-d}}(k)$ induces an action of $PSL_2(\mathscr{O}_{-d})$ provided k is even.

```
──────────────────────── Example ────────────────────────
gap> R:=ResolutionPSL2QuadraticIntegers(-6,4);
Resolution of length 4 in characteristic 0 for PSL(2,0-6) .
No contracting homotopy available.

gap> G:=R!.group;;
gap> M:=HomogeneousPolynomials(G,24);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,1);
[ 2, 2, 2, 2, 12, 24, 9240, 55440, 0, 0, 0, 0 ]
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 6, 6, 6, 6, 6, 6, 6, 6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 24, 120, 120,
  840, 840, 840, 2520, 27720, 27720, 24227280, 24227280, 411863760, 411863760,
  245443824374892865187742514283666449812984 0,
```

```
    14726629462493571911264550857019986988779040, 0, 0, 0, 0 ]
gap> Cohomology(C,3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 12,
  12 ]
```

We can also consider the coefficient module

$$P_{\mathcal{O}_{-d}}(k,\ell) = P_{\mathcal{O}_{-d}}(k) \otimes_{\mathcal{O}_{-d}} \overline{P_{\mathcal{O}_{-d}}(\ell)}$$

where the bar denotes a twist in the action obtained from complex conjugation. For an action of the projective linear group we must insist that $k+\ell$ is even. The following commands compute

$$H^2(PSL_2(\mathcal{O}_{-11}), P_{\mathcal{O}_{-11}}(5,5)) = (\mathbb{Z}_2)^8 \oplus \mathbb{Z}_{60} \oplus (\mathbb{Z}_{660})^3 \oplus \mathbb{Z}^6,$$

a computation which was first made, along with many other cohomology computationsfor Bianchi groups, by Mehmet Haluk Sengun [Sen11].

```
                          ─────────────── Example ───────────────
  gap> R:=ResolutionPSL2QuadraticIntegers(-11,3);;
  gap> M:=HomogeneousPolynomials(R!.group,5,5);;
  gap> C:=HomToIntegralModule(R,M);;
  gap> Cohomology(C,2);
  [ 2, 2, 2, 2, 2, 2, 2, 2, 60, 660, 660, 660, 0, 0, 0, 0, 0, 0 ]
```

The function `ResolutionPSL2QuadraticIntegers(-d,n)` relies on a limited data base produced by the algorithms implemented by Schoennenbeck and Rahm. The function also covers some cases covered by entering a sring "-d+I" as first variable. These cases correspond to projective special groups of module automorphisms of lattices of rank 2 over the integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ with non-trivial Steinitz-class. In the case of a larger class group there are cases labelled "-d+I2",...,"-d+Ik" and the Ij together with O-d form a system of representatives of elements of the class group modulo squares and Galois action. For instance, the following commands compute

$$H_2(PSL(\mathcal{O}_{-21+I2}), \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}^6.$$

```
                          ─────────────── Example ───────────────
  gap> R:=ResolutionPSL2QuadraticIntegers("-21+I2",3);
  Resolution of length 3 in characteristic 0 for PSL(2,O-21+I2)) .
  No contracting homotopy available.

  gap> Homology(TensorWithIntegers(R),2);
  [ 2, 0, 0, 0, 0, 0, 0 ]
```

13.11 Some other infinite matrix groups

Analogous to the functions for Bianchi groups, HAP has functions

- `ResolutionSL2QuadraticIntegers(-d,n)`

- `ResolutionSL2ZInvertedInteger(m,n)`

- `ResolutionGL2QuadraticIntegers(-d,n)`

- `ResolutionPGL2QuadraticIntegers(-d,n)`

- `ResolutionGL3QuadraticIntegers(-d,n)`

- `ResolutionPGL3QuadraticIntegers(-d,n)`

for computing free resolutions for certain values of $SL_2(\mathscr{O}_{-d})$, $SL_2(\mathbb{Z}[\frac{1}{m}])$, $GL_2(\mathscr{O}_{-d})$ and $PGL_2(\mathscr{O}_{-d})$. Additionally, the function

- `ResolutionArithmeticGroup("string",n)`

can be used to compute resolutions for groups whose data (provided by Sebastian Schoennenbeck, Alexander Rahm and Mathieu Dutour) is stored in the directory `gap/pkg/Hap/lib/Perturbations/Gcomplexes` .

For instance, the following commands compute

$$H^1(SL_2(\mathscr{O}_{-6}), P_{\mathscr{O}_{-6}}(24)) = (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{9240} \oplus \mathbb{Z}_{55440} \oplus \mathbb{Z}^4,$$

$$H^2(SL_2(\mathscr{O}_{-6}), P_{\mathscr{O}_{-6}}(24)) = \begin{array}{l} (\mathbb{Z}_2)^{26} \oplus (Z_6)^7 \oplus (Z_{12})^{10} \oplus \mathbb{Z}_{24} \oplus (\mathbb{Z}_{120})^2 \oplus (\mathbb{Z}_{840})^3 \\ \oplus \mathbb{Z}_{2520} \oplus (\mathbb{Z}_{27720})^2 \oplus (\mathbb{Z}_{24227280})^2 \oplus (\mathbb{Z}_{411863760})^2 \\ \oplus \mathbb{Z}_{2454438243748928651877425142836664498129840} \\ \oplus \mathbb{Z}_{1472662946249357191126455085701998698779040} \\ \oplus \mathbb{Z}^4 \end{array},$$

$$H^3(SL_2(\mathscr{O}_{-6}), P_{\mathscr{O}_{-6}}(24)) = (\mathbb{Z}_2)^{58} \oplus (\mathbb{Z}_4)^4 \oplus (\mathbb{Z}_{12}).$$

```
────────────────────── Example ──────────────────────
gap> R:=ResolutionSL2QuadraticIntegers(-6,4);
Resolution of length 4 in characteristic 0 for PSL(2,O-6) .
No contracting homotopy available.

gap> G:=R!.group;;
gap> M:=HomogeneousPolynomials(G,24);;
gap> C:=HomToIntegralModule(R,M);;
gap> Cohomology(C,1);
[ 2, 2, 2, 2, 12, 24, 9240, 55440, 0, 0, 0, 0 ]
gap> Cohomology(C,2);
gap> Cohomology(C,2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 6, 6, 6, 6, 6, 6, 6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 24, 120,
  120, 840, 840, 840, 2520, 27720, 27720, 24227280, 24227280, 411863760,
  411863760, 2454438243748928651877425142836664498129840,
  1472662946249357191126455085701998698779040, 0, 0, 0, 0 ]
gap> Cohomology(C,3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 12, 12 ]
```

The following commands construct free resolutions up to degree 5 for the groups $SL_2(\mathbb{Z}[\frac{1}{2}])$, $GL_2(\mathcal{O}_{-2})$, $GL_2(\mathcal{O}_2)$, $PGL_2(\mathcal{O}_2)$, $GL_3(\mathcal{O}_{-2})$, $PGL_3(\mathcal{O}_{-2})$. The final command constructs a free resolution up to degree 3 for $PSL_4(\mathbb{Z})$.

```
                              Example
gap> R1:=ResolutionSL2ZInvertedInteger(2,5);
Resolution of length 5 in characteristic 0 for SL(2,Z[1/2]) .

gap> R2:=ResolutionGL2QuadraticIntegers(-2,5);
Resolution of length 5 in characteristic 0 for GL(2,O-2) .
No contracting homotopy available.

gap> R3:=ResolutionGL2QuadraticIntegers(2,5);
Resolution of length 5 in characteristic 0 for GL(2,O2) .
No contracting homotopy available.

gap> R4:=ResolutionPGL2QuadraticIntegers(2,5);
Resolution of length 5 in characteristic 0 for PGL(2,O2) .
No contracting homotopy available.

gap> R5:=ResolutionGL3QuadraticIntegers(-2,5);
Resolution of length 5 in characteristic 0 for GL(3,O-2) .
No contracting homotopy available.

gap> R6:=ResolutionPGL3QuadraticIntegers(-2,5);
Resolution of length 5 in characteristic 0 for PGL(3,O-2) .
No contracting homotopy available.

gap> R7:=ResolutionArithmeticGroup("PSL(4,Z)",3);
Resolution of length 3 in characteristic 0 for <matrix group with 655 generators> .
No contracting homotopy available.
```

13.12   Ideals and finite quotient groups

The following commands first construct the number field $\mathbb{Q}(\sqrt{-7})$, its ring of integers $\mathcal{O}_{-7} = \mathcal{O}(\mathbb{Q}(\sqrt{-7}))$, and the principal ideal $I = \langle 5 + 2\sqrt{-7} \rangle \lhd \mathcal{O}(\mathbb{Q}(\sqrt{-7}))$ of norm $\mathcal{N}(I) = 53$. The ring I is prime since its norm is a prime number. The primality of I is also demonstrated by observing that the quotient ring $R = \mathcal{O}_{-7}/I$ is an integral domain and hence isomorphic to the unique finite field of order 53, $R \cong \mathbb{Z}/53\mathbb{Z}$. (In a ring of quadratic integers *prime ideal* is the same as *maximal ideal*).

The finite group $G = SL_2(\mathcal{O}_{-7}/I)$ is then constructed and confirmed to be isomorphic to $SL_2(\mathbb{Z}/53\mathbb{Z})$. The group G is shown to admit a periodic $\mathbb{Z}G$-resolution of $\mathbb{Z}$ of period dividing 52.

Finally the integral homology

$$H_n(G, \mathbb{Z}) = \begin{cases} 0 & n \neq 3, 7, \text{ for } 0 \leq n \leq 8, \\ \mathbb{Z}_{2808} & n = 3, 7, \end{cases}$$

is computed.

```
                              Example
gap> Q:=QuadraticNumberField(-7);
Q(Sqrt(-7))
```

```
gap> OQ:=RingOfIntegers(Q);
O(Q(Sqrt(-7)))

gap> I:=QuadraticIdeal(OQ,5+2*Sqrt(-7));
ideal of norm 53 in O(Q(Sqrt(-7)))

gap> R:=OQ mod I;
ring mod ideal of norm 53

gap> IsIntegralRing(R);
true

gap> gens:=GeneratorsOfGroup( SL2QuadraticIntegers(-7) );;
gap> G:=Group(gens*One(R));;G:=Image(IsomorphismPermGroup(G));;
gap> StructureDescription(G);
"SL(2,53)"

gap> IsPeriodic(G);
true
gap> CohomologicalPeriod(G);
52

gap> GroupHomology(G,1);
[  ]
gap> GroupHomology(G,2);
[  ]
gap> GroupHomology(G,3);
[ 8, 27, 13 ]
gap> GroupHomology(G,4);
[  ]
gap> GroupHomology(G,5);
[  ]
gap> GroupHomology(G,6);
[  ]
gap> GroupHomology(G,7);
[ 8, 27, 13 ]
gap> GroupHomology(G,8);
[  ]
```

The following commands show that the rational prime 7 is not prime in $\mathscr{O}_{-5} = \mathscr{O}(\mathbb{Q}(\sqrt{-5}))$. Moreover, 7 totally splits in $\mathscr{O}_{-5}$ since the final command shows that only the rational primes 2 and 5 ramify in $\mathscr{O}_{-5}$.

──────── Example ────────

```
gap> Q:=QuadraticNumberField(-5);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,7);;
gap> IsPrime(I);
false

gap> Factors(Discriminant(OQ));
```

```
[ -2, 2, 5 ]
```

For $d < 0$ the rings $\mathscr{O}_d = \mathscr{O}(\mathbb{Q}(\sqrt{d}))$ are unique factorization domains for precisely

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163.$$

This result was conjectured by Gauss, and essentially proved by Kurt Heegner, and then later proved by Harold Stark.

The following commands construct the classic example of a prime ideal I that is not principal. They then illustrate reduction modulo I.

```
———————————————————— Example ————————————————————
gap> Q:=QuadraticNumberField(-5);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,[2,1+Sqrt(-5)]);
ideal of norm 2 in O(Q(Sqrt(-5)))

gap> 6 mod I;
0
```

13.13   Congruence subgroups for ideals

Given a ring of integers $\mathscr{O}$ and ideal $I \lhd \mathscr{O}$ there is a canonical homomorphism $\pi_I : SL_2(\mathscr{O}) \to SL_2(\mathscr{O}/I)$. A subgroup $\Gamma \le SL_2(\mathscr{O})$ is said to be a *congruence subgroup* if it contains $\ker \pi_I$. Thus congruence subgroups are of finite index. Generalizing the definition in 13.1 above, we define the *principal congruence subgroup* $\Gamma_1(I) = \ker \pi_I$, and the congruence subgroup $\Gamma_0(I)$ consisting of preimages of the upper triangular matrices in $SL_2(\mathscr{O}/I)$.

The following commands construct $\Gamma = \Gamma_0(I)$ for the ideal $I \lhd \mathscr{O}\mathbb{Q}(\sqrt{-5})$ generated by 12 and $36\sqrt{-5}$. The group Γ has index 385 in $SL_2(\mathscr{O}\mathbb{Q}(\sqrt{-5}))$. The final command displays a tree in a Cayley graph for $SL_2(\mathscr{O}\mathbb{Q}(\sqrt{-5}))$ whose nodes represent a transversal for Γ.

```
———————————————————— Example ————————————————————
gap> Q:=QuadraticNumberField(-5);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,[36*Sqrt(-5), 12]);;
gap> G:=HAP_CongruenceSubgroupGamma0(I);
CongruenceSubgroupGamma0(ideal of norm 144 in O(Q(Sqrt(-5))))

gap> IndexInSL2O(G);
385

gap> HAP_SL2TreeDisplay(G);
```

The next commands first construct the congruence subgroup $\Gamma_0(I)$ of index 144 in $SL_2(\mathscr{O}\mathbb{Q}(\sqrt{-2}))$ for the ideal I in $\mathscr{O}\mathbb{Q}(\sqrt{-2})$ generated by $4 + 5\sqrt{-2}$. The commands then compute

$$H_1(\Gamma_0(I), \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}^8,$$

$$H_2(\Gamma_0(I),\mathbb{Z}) = (\mathbb{Z}_2)^9 \oplus \mathbb{Z}^7,$$

$$H_3(\Gamma_0(I),\mathbb{Z}) = (\mathbb{Z}_2)^9.$$

———————— Example ————————
```
gap> Q:=QuadraticNumberField(-2);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,4+5*Sqrt(-2));;
gap> G:=HAP_CongruenceSubgroupGamma0(I);
CongruenceSubgroupGamma0(ideal of norm 66 in O(Q(Sqrt(-2))))

gap> IndexInSL2O(G);
144

gap> R:=ResolutionSL2QuadraticIntegers(-2,4,true);;
gap> S:=ResolutionFiniteSubgroup(R,G);;

gap> Homology(TensorWithIntegers(S),1);
[ 3, 6, 30, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Homology(TensorWithIntegers(S),2);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0 ]
gap> Homology(TensorWithIntegers(S),3);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

13.14 First homology

The isomorphism $H_1(G,\mathbb{Z}) \cong G_{ab}$ allows for the computation of first integral homology using computational methods for finitely presented groups. Such methods underly the following computation of

$$H_1(\Gamma_0(I),\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_{4078793513671}$$

where I is the prime ideal in the Gaussian integers generated by $41 + 56\sqrt{-1}$.

———————— Example ————————
```
gap> Q:=QuadraticNumberField(-1);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,41+56*Sqrt(-1));
ideal of norm 4817 in O(GaussianRationals)
gap> G:=HAP_CongruenceSubgroupGamma0(I);;
gap> AbelianInvariants(G);
[ 2, 2, 4, 5, 7, 16, 29, 43, 157, 179, 1877, 7741, 22037, 292306033,
  4078793513671 ]
```

We write G_{tors}^{ab} to denote the maximal finite summand of the first homology group of G and refer to this as the *torsion subgroup*. Nicholas Bergeron and Akshay Venkatesh [Ber16] have conjectured

relationships between the torsion in congruence subgroups Γ and the volume of their quotient manifold $\mathfrak{h}^3/\Gamma$. For instance, for the Gaussian integers they conjecture

$$\frac{\log|\Gamma_0(I)^{ab}_{tors}|}{\mathrm{Norm}(I)} \rightarrow \frac{\lambda}{18\pi}, \ \lambda = L(2, \chi_{\mathbb{Q}(\sqrt{-1})}) = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots$$

as the norm of the prime ideal I tends to ∞. The following approximates $\lambda/18\pi = 0.0161957$ and $\frac{\log|\Gamma_0(I)^{ab}_{tors}|}{\mathrm{Norm}(I)} = 0.00913432$ for the above example.

```
――――――――――――――― Example ―――――――――――――――
gap> Q:=QuadraticNumberField(-1);;
gap> Lfunction(Q,2)/(18*3.142);
0.0161957

gap> 1.0*Log(Product(AbelianInvariants(F)),10)/Norm(I);
0.00913432
```

The link with volume is given by the Humbert volume formula

$$\mathrm{Vol}(\mathfrak{h}^3/PSL_2(\mathcal{O}_d)) = \frac{|D|^{3/2}}{24} \zeta_{\mathbb{Q}(\sqrt{d})}(2)/\zeta_{\mathbb{Q}}(2)$$

valid for square-free $d < 0$, where D is the discriminant of $\mathbb{Q}(\sqrt{d})$. The volume of a finite index subgroup Γ is obtained by multiplying the right-hand side by the index $|PSL_2(\mathcal{O}_d) : \Gamma|$.

Chapter 14

Parallel computation

14.1 An embarassingly parallel computation

The following example creates five child processes and uses them simultaneously to compute the second integral homology of each of the 267 groups of order 64. The final command shows that

$$H_2(G,\mathbb{Z}) = \mathbb{Z}_2^{15}$$

for the 267-th group G in GAP's library of small groups.

```
──────────────────────────────── Example ────────────────────────────────
gap> Processes:=List([1..5],i->ChildProcess());;
gap> fn:=function(i);return GroupHomology(SmallGroup(64,i),2);end;;
gap> for p in Processes do
> ChildPut(fn,"fn",p);
> od;

gap> NrSmallGroups(64);
267

gap> L:=ParallelList([1..267],"fn",Processes);;

gap> L[267];
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
```

The function `ParallelList()` is built from HAP's six core functions for parallel computation.

14.2 An non-embarassingly parallel computation

The following commands use core functions to compute the product $A = M \times N$ of two random matrices by distributing the work over two processors.

```
──────────────────────────────── Example ────────────────────────────────
gap> M:=RandomMat(2000,2000);;
gap> N:=RandomMat(2000,2000);;

gap> s:=ChildProcess();;

gap> Mtop:=M{[1..1000]};;
gap> Mbottom:=M{[1001..2000]};;
```

```
gap> ChildPut(Mtop,"Mtop",s);
gap> ChildPut(N,"N",s);
gap> NextAvailableChild([s]);;

gap> ChildCommand("Atop:=Mtop*N;;",s);;
gap> Abottom:=Mbottom*N;;
gap> A:=ChildGet("Atop",s);;
gap> Append(A,Abottom);;
```

Chapter 15

Regular CW-structure on knots

15.1   Knot complements in the 3-ball

While methods for endowing knot complements with CW-structure already exist in HAP (see section 2.1), they often result in a large number of cells which can make computing with them taxing. The following example shows how one can obtain a comparatively small 3-dimensional regular CW-complex corresponding to the complement of a thickened trefoil knot from an arc presentation. Recall that an arc presentation is encoded in HAP as a list of integer pairs corresponding to the position of the endpoints of each horizontal arc in a grid.

```
───────────────────────────── Example ─────────────────────────────
  gap> k_:=PureCubicalKnot(3,1);
  prime knot 1 with 3 crossings

  gap> arc:=ArcPresentation(k_);
  [ [ 2, 5 ], [ 1, 3 ], [ 2, 4 ], [ 3, 5 ], [ 1, 4 ] ]
  gap> k_:=RegularCWComplex(PureComplexComplement(k_));
  Regular CW-complex of dimension 3

  gap> Size(k_);
  13291
  gap> k:=KnotComplement(arc);
  Regular CW-complex of dimension 3

  gap> Size(k);
  395
```

An optional argument of `"rand"` in the `KnotComplement` function randomises the order in which 2-cells are added to the complex. This allows for alternate presentations of the knot group.

```
───────────────────────────── Example ─────────────────────────────
  gap> arc:=ArcPresentation(PureCubicalKnot(3,1));
  [ [ 2, 5 ], [ 1, 3 ], [ 2, 4 ], [ 3, 5 ], [ 1, 4 ] ]
  gap> k:=KnotComplement(arc,"rand");
  Random 2-cell selection is enabled.
  Regular CW-complex of dimension 3

  gap> g:=FundamentalGroup(k); RelatorsOfFpGroup(g);
```

```
#I  there are 2 generators and 1 relator of total length 6
<fp group of size infinity on the generators [ f1, f2 ]>
[ f2^-1*f1*f2^-1*f1^-1*f2*f1^-1 ]
gap> k:=KnotComplement(arc,"rand");
Random 2-cell selection is enabled.
Regular CW-complex of dimension 3

gap> g:=FundamentalGroup(k); RelatorsOfFpGroup(g);
#I  there are 2 generators and 1 relator of total length 7
<fp group of size infinity on the generators [ f1, f2 ]>
[ f1*f2^-2*f1*f2*f1^-1*f2 ]
```

It is often useful to obtain an inclusion of regular CW-complexes $\iota : \partial(N(K)) \hookrightarrow B^3 \setminus N(K)$ from the boundary of a tubular neighbourhood of some knot $N(K)$ into its complement in the 3-ball $B^3 \setminus N(K)$. The below example does this for the first prime knot on 11 crossings.

```
—————————————————— Example ——————————————————
gap> arc:=ArcPresentation(PureCubicalKnot(11,1));
[ [ 2, 9 ], [ 1, 3 ], [ 2, 6 ], [ 4, 7 ], [ 3, 5 ], [ 6, 10 ], [ 4, 8 ],
  [ 9, 11 ], [ 7, 10 ], [ 1, 8 ], [ 5, 11 ] ]
gap> k:=KnotComplementWithBoundary(arc);
Map of regular CW-complexes

gap> Size(Source(i));
616
gap> Size(Target(i));
1043
```

Note that we can add n-cells to regular CW-complexes by specifying the $(n-1)$-cells in their boundaries and $(n+1)$-cells in their coboundaries.

```
—————————————————— Example ——————————————————
gap> k:=KnotComplement([[1,2],[1,2]])!.boundaries;;
gap> Homology(RegularCWComplex(k),0);
[ 0 ]
gap> AddCell(k,0,[0],[]);
gap> Homology(RegularCWComplex(k),0);
[ 0, 0 ]
```

15.2 Tubular neighbourhoods

Let Y denote a CW-subcomplex of a regular CW-complex X and let $N(Y)$ denote an open tubular neighbourhood of Y. Given an inclusion of regular CW-complexes $f : Y \hookrightarrow X$, this algorithm describes a procedure for obtaining the associated inclusion $f' : \partial C \hookrightarrow C$ where $C = X \setminus N(Y)$ and ∂C denotes the boundary of C. The following is also assumed:

Let e^n denote a cell of $X \setminus Y$ of dimension n with $\bar{e}^n$ denoting its closure. For each n-cell, all of the connected components of the subcomplex $\bar{e}^n \cap Y$ are contractible.

Some additional terminology and notation is needed to describe this algorithm. The output regular CW-complex $X \setminus N(Y)$ consists of the cell complex $X \setminus Y$ as well as some additional cells to maintain

regularity. A cell of $X \setminus N(Y)$ is referred to as *internal* if it lies in $X \setminus Y$, it is *external* otherwise. Let $\bar{e}^n$ denote the closure in X of an internal cell e^n. Note that $\bar{e}^n$ is a CW-subcomplex of X and so is the intersection $\bar{e}^n \cap Y$ which can be expressed as the union

$$\bar{e}^n \cap Y = A_1 \cup A_2 \cup \cdots \cup A_k$$

of its path components A_i all of which are CW-subcomplexes of Y. For each n-cell of $X \setminus Y$ there is one internal n-cell e^n of $X \setminus N(Y)$. For $n \geq 1$ there is also one external $(n-1)$-cell $f_{A_i}^{e^n}$ for each path component A_i of $\bar{e}^n \cap Y$. Lastly, we need a method for determining the homological boundary of the internal and external cells:

- The boundary of an internal n-cell e^n consists of all those internal $(n-1)$-cells of $\bar{e}^n$ together with all external $(n-1)$-cells $f_{A_i}^{e^n}$ where A_i is a path component of $\bar{e}^n \cap Y$.

- The boundary of an external $(n-1)$-cell $f_{A_i}^{e^n}$ consists of all those external $(n-2)$-cells $f_{B_j}^{e^{n-1}}$ where e^{n-1} is an $(n-1)$-cell of $\bar{e}^n$ and $B_j \subseteq A_i$ is a path component of A_i.

The following three steps comprise the algorithm.

(1) For each internal n-cell $e^n \subset X \setminus Y$, compute the CW-complex $\bar{e}^n \cap Y$ as a union of path components $A_1 \cup A_2 \cup \cdots \cup A_k$. This information can be used to determine the number of cells of $X \setminus N(Y)$ in each dimension.

(2) Create a list $B = [\,[\ ],[\ ],\ldots,[\ ]\,]$ of length $\dim X + 1$.

(3) For $0 \leq n \leq \dim X$ set $B[n+1] = [b_1, b_2, \ldots, b_{\alpha_n}]$ where α_n is the number of n-cells in $X \setminus N(Y)$ and b_i is a list of integers describing the $(n-1)$-cells of the ith n-cell of $X \setminus N(Y)$. The internal cells will always be listed before the external cells in each sublist. Return B as a regular CW-complex.

The following example computes the tubular neighbourhood of a 1-dimensional subcomplex of a 3-dimensional complex corresponding to the Hopf link embedded in the closed 3-ball.

```
─────── Example ───────
gap> arc:=[[2,4],[1,3],[2,4],[1,3]];;
gap> f:=ArcPresentationToKnottedOneComplex(arc);
Map of regular CW-complexes

gap> comp:=RegularCWComplexComplement(f);
Testing contractibility...
151 out of 151 cells tested.
The input is compatible with this algorithm.
Regular CW-complex of dimension 3
```

Note that the output of this algorithm is just a regular CW-complex, not an inclusion map. The function `BoundaryMap` can be employed to obtain the boundary of a pure complex. This results in three path components for this example: two corresponding to the boundary of the knotted tori and the other corresponding to the boundary of the 3-ball in which the link was embedded. These path components can be obtained as individual CW-subcomplexes if desired. A CW-subcomplex is represented in HAP as a list $[X, s]$ where X is a regular CW-complex and s is a list of length n whose i^{th} entry lists the indexing of each $(i-1)$-cell of the n-dimensional subcomplex of X. CW-subcomplexes and CW maps can be converted between each other interchangeably. This next example obtains the inclusion detailed in the above algorithm, finds the path components of the source of said inclusion, shows that they are in fact disjoint, and then obtains the first four integral homology groups of each component.

```
─────── Example ───────
gap> f_:=BoundaryMap(comp);
Map of regular CW-complexes
```

```
gap> f_:=RegularCWMapToCWSubcomplex(f_);;
gap> paths:=PathComponentsCWSubcomplex(f_);
[ [ Regular CW-complex of dimension 3
        ,
        [ [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 18, 19, 20 ],
          [ 1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18, 33, 34, 35, 46, 47, 48
            ], [ 11, 12, 13, 14, 15, 16, 35, 36 ] ] ],
  [ Regular CW-complex of dimension 3
        , [ [ 21, 24, 25, 27, 30, 31, 32, 37, 38, 39, 40, 43, 45, 46, 48 ],
          [ 49, 51, 53, 56, 57, 59, 61, 63, 65, 67, 69, 71, 73, 74, 76, 79,
              82, 83, 86, 87, 90, 91 ], [ 37, 39, 41, 44, 45, 47, 49 ] ] ],
  [ Regular CW-complex of dimension 3
        , [ [ 22, 23, 26, 28, 29, 33, 34, 35, 36, 41, 42, 44, 47, 49, 50 ],
          [ 50, 52, 54, 55, 58, 60, 62, 64, 66, 68, 70, 72, 75, 77, 78, 80,
              81, 84, 85, 88, 89, 92 ], [ 38, 40, 42, 43, 46, 48, 50 ] ] ] ]
gap> paths:=List(paths,CWSubcomplexToRegularCWMap);
[ Map of regular CW-complexes
    , Map of regular CW-complexes
    , Map of regular CW-complexes
    ]
gap> List([1..3],x->List(Difference([1..3],[x]),y->IntersectionCWSubcomplex(paths[x],paths[y])));
[ [ [ Regular CW-complex of dimension 3
          , [ [  ], [  ], [  ] ] ], [ Regular CW-complex of dimension 3
          , [ [  ], [  ], [  ] ] ] ], [ [ Regular CW-complex of dimension 3
          , [ [  ], [  ], [  ] ] ], [ Regular CW-complex of dimension 3
          , [ [  ], [  ], [  ] ] ] ], [ [ Regular CW-complex of dimension 3
          , [ [  ], [  ], [  ] ] ], [ Regular CW-complex of dimension 3
          , [ [  ], [  ], [  ] ] ] ] ]

gap> List(paths,x->List([0..3],y->Homology(Source(x),y)));
[ [ [ 0 ], [  ], [ 0 ], [  ] ], [ [ 0 ], [ 0, 0 ], [ 0 ], [  ] ],
  [ [ 0 ], [ 0, 0 ], [ 0 ], [  ] ] ]
```

As previously mentioned, for the tubular neighbourhood algorithm to work, we require that no external cells yield non-contractible path-components in their intersection with the subcomplex. If this is ever the case then we can subdivide the offending cell to prevent this from happening. We have implemented two subdivision algorithms in HAP, one for barycentrically subdividing a given cell, and the other for subdividing an n-cell into as many n-cells as there are $(n-1)$-cells in its boundary. Barycentric subdivision is integrated into the `RegularCWComplexComplement` function and will be performed automatically as required. The following example shows this automatic subdivision running via the complement of a tubular neighbourhood of the unknot, then obtains an inclusion map from the closure of an arbitrary 3-cell of this complex and then compares the difference in size of the two different subdivisions of a 2-cell in the boundary of this 3-cell.

```
―――――――――――――――――――――― Example ――――――――――――――――――――――
gap> arc:=[[1,2],[1,2]];;
gap> unknot:=ArcPresentationToKnottedOneComplex(arc);
Map of regular CW-complexes

gap> f:=RegularCWComplexComplement(unknot);
```

```
Testing contractibility...
79 out of 79 cells tested.
Subdividing 3 cell(s):
100% complete.
Testing contractibility...
145 out of 145 cells tested.
The input is compatible with this algorithm.
Regular CW-complex of dimension 3

gap> f:=Objectify(HapRegularCWMap,rec(source:=f,target:=f,mapping:={i,j}->j));
Map of regular CW-complexes

gap> closure:=ClosureCWCell(Target(f),3,1);
[ Regular CW-complex of dimension 3

    ,
  [ [ 1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 14, 20, 21, 22, 23, 25 ],
      [ 1, 2, 3, 7, 8, 9, 10, 11, 15, 16, 17, 20, 21, 22, 23, 24, 25, 27, 28, 55, 58, 59,
          60, 63 ], [ 1, 4, 7, 8, 9, 13, 14, 15, 18, 52 ], [ 1 ] ] ]
gap> Size(Target(f));
195
gap> Size(Target(BarycentricallySubdivideCell(f,2,1)));
231
gap> Size(Target(SubdivideCell(f,2,1)));
207
```

15.3   Knotted surface complements in the 4-ball

A construction of Satoh's, the tube map, associates a ribbon torus-knot to virtual knot diagrams. A virtual knot diagram differs from a knot diagram in that it allows for a third type of crossing, a virtual crossing. The image of such a crossing via the tube map is two tori which pass through each other. An arc diagram is a triple of lists [arc,cross,cols] that encode virtual knot diagrams. arc is an arc presentation. cross is a list of length the number of crossings in the knot associated to the arc presentation whose entries are $-1, 0$ or 1 corresponding to an undercrossing (horizontal arc underneath vertical arc), a virtual crossing (depicted by intersecting horizontal and vertical arcs) and an overcrossing (horizontal arc above vertical arc) respectively. cols is a list of length the number of 0 entries in cross and its entries are $1, 2, 3$ or 4. It describes the types of 'colourings' we assign to the virtual crossings. We interpret each integer as the change in 4-dimensional height information as represented by a colour scale from blue (lower down in 4-space), to green (0 level), to red (higher up in 4-space). Without loss of generality, we impose that at each virtual crossing, the vertical arc passes through the horizontal arc. Thus, 1 corresponds to the vertical bar entering the horizontal bar as blue and leaving as blue, 2 corresponds to entering as blue and leaving as red, 3 corresponds to entering as red and leaving as blue and 4 corresponds to entering and leaving as red. A coloured arc diagram can be visualised using the ViewColouredArcDiagram function.

```
────────────────── Example ──────────────────
gap> arc:=ArcPresentation(PureCubicalKnot(6,1));
[ [ 5, 8 ], [ 4, 6 ], [ 3, 5 ], [ 2, 4 ], [ 1, 3 ], [ 2, 7 ], [ 6, 8 ], [ 1, 7 ] ]
gap> cross:=[0,0,1,-1,-1,0];;
gap> cols:=[1,4,3];;
```

```
gap> ViewColouredArcDiagram(arc,cross,cols);
convert-im6.q16: pixels are not authentic '/tmp/HAPtmpImage.txt' @ error/cache.c/QueueAuthenticPi
```

Towards obtaining a regular CW-decomposition of ribbon torus-knots, we first begin by embedding a self-intersecting knotted torus in the 3-ball. The function `ArcDiagramToTubularSurface` inputs a coloured arc diagram and outputs an inclusion from the boundary of some (potentially self-intersecting) torus in the 3-ball. By inputting just an arc presentation, one can obtain an inclusion identical to the `KnotComplementWithBoundary` function. By additionally inputting a list of -1s and 1s, one can obtain an inclusion similar to `KnotComplementWithBoundary` but where there is extra freedom in determining whether or not a given crossing is an under/overcrossing. If one inputs both of the above but includes 0 entries in the `cross` list and includes the list of colours, the output is then an inclusion from an embedded self-intersecting torus into the 3-ball where each 2-cell (the top-dimensional cells of the self-intersecting surface) is assigned a colour.

```
───────────────────────────── Example ─────────────────────────────
gap> tub:=ArcDiagramToTubularSurface(arc);
Map of regular CW-complexes

gap> tub:=ArcDiagramToTubularSurface([arc,cross]);
Map of regular CW-complexes

gap> tub:=ArcDiagramToTubularSurface([arc,cross,cols]);
Map of regular CW-complexes

gap> List([1..Length(Source(tub)!.boundaries[3])],x->tub!.colour(2,tub!.mapping(2,x)));
[ [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ],
  [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ],
  [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ],
  [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ],
  [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ -1 ], [ -1 ],
  [ 0 ], [ 0 ], [ -1 ], [ -1 ], [ -1 ], [ -1 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 1 ], [ 1 ],
  [ 0 ], [ 0 ], [ 1 ], [ 1 ], [ 1 ], [ 1 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 1 ], [ 1 ], [ 0 ],
  [ 0 ], [ -1 ], [ -1 ], [ 1 ], [ -1 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ -1 ], [ -1 ], [ 0 ],
  [ 1 ], [ 1 ], [ 0 ], [ 0 ], [ 0 ], [ 0 ], [ 1 ], [ -1 ], [ 0 ] ]
```

From this self-intersecting surface with colour, we can lift it to a surface without self-intersections in $\mathbb{R}^4$. We do this by constructing a regular CW-complex of the direct product $B^3 \times [a,b]$ where B^3 denotes the 3-ball, a is 1 less than the smallest integer assigned to a cell by the colouring, and b is 1 greater than the largest integer assigned to a cell by the colouring. The subcomplex of the direct product corresponding to the surface without intersection can be obtained using the colouring with additional care taken to not lift any 1-cells arising as double-point singularities. The following example constructs the complement of a ribbon torus-link embedded in $\mathbb{R}^4$ obtained from the Hopf link with one virtual crossing and then calculates some invariants of the resulting space. We compare the size of this complex, as well as how long it takes to obtain the same invariants, with a cubical complex of the same space. As barycentric subdivision can massively increase the size of the cell complex, the below method sequentially obtains the tubular neighbourhood of the entire subcomplex by obtaining the tubular neighbourhood of each individual 2-cell. This has yet to be optimised so it currently takes some time to complete.

---- Example ----

```
gap> arc:=[[2,4],[1,3],[2,4],[1,3]];;
gap> tub:=ArcDiagramToTubularSurface([arc,[0,-1],[2]]);
Map of regular CW-complexes

gap> tub:=LiftColouredSurface(tub);
Map of regular CW-complexes

gap> Dimension(Source(tub));
2
gap> Dimension(Source(tub));
4
gap> map:=RegularCWMapToCWSubcomplex(tub);;
gap> sub:=SortedList(map[2][3]);;
gap> sub:=List(sub,x->x-(Position(sub,x)-1));;
gap> clsr:=ClosureCWCell(map[1],2,sub[1])[2];;
gap> seq:=CWSubcomplexToRegularCWMap([map[1],clsr]);;
gap> tub:=RegularCWComplexComplement(seq);
Testing contractibility...
3501 out of 3501 cells tested.
The input is compatible with this algorithm.
gap> for i in [2..Length(sub)] do
>     clsr:=ClosureCWCell(tub,2,sub[i])[2];;
>     seq:=CWSubcomplexToRegularCWMap([tub,clsr]);;
>     tub:=RegularCWComplexComplement(seq);
> od;
Testing contractibility...
3612 out of 3612 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
3693 out of 3693 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
3871 out of 3871 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
3925 out of 3925 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
4084 out of 4084 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
4216 out of 4216 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
4348 out of 4348 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
4529 out of 4529 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
4688 out of 4688 cells tested.
The input is compatible with this algorithm.
```

```
Testing contractibility...
4723 out of 4723 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
4918 out of 4918 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
5107 out of 5107 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
5269 out of 5269 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
5401 out of 5401 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
5548 out of 5548 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
5702 out of 5702 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
5846 out of 5846 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6027 out of 6027 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6089 out of 6089 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6124 out of 6124 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6159 out of 6159 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6349 out of 6349 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6467 out of 6467 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6639 out of 6639 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6757 out of 6757 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
6962 out of 6962 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
```

```
7052 out of 7052 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7242 out of 7242 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7360 out of 7360 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7470 out of 7470 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7561 out of 7561 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7624 out of 7624 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7764 out of 7764 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7904 out of 7904 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
7979 out of 7979 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8024 out of 8024 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8086 out of 8086 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8148 out of 8148 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8202 out of 8202 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8396 out of 8396 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8534 out of 8534 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8625 out of 8625 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8736 out of 8736 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
8817 out of 8817 cells tested.
```

```
The input is compatible with this algorithm.
Testing contractibility...
8983 out of 8983 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9073 out of 9073 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9218 out of 9218 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9323 out of 9323 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9442 out of 9442 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9487 out of 9487 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9538 out of 9538 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9583 out of 9583 cells tested.
The input is compatible with this algorithm.
Testing contractibility...
9634 out of 9634 cells tested.
The input is compatible with this algorithm.
gap> Size(tub);
9685
gap> total_time_1:=0;;
gap> List([0..4],x->Homology(tub,x)); total_time_1:=total_time_1+time;;
[ [ 0 ], [ 0, 0 ], [ 0, 0, 0, 0 ], [ 0, 0 ], [ ] ]
gap> c:=ChainComplexOfUniversalCover(tub);; total_time_1:=total_time_1+time;;
gap> l:=Filtered(LowIndexSubgroups(c!.group,5),g->Index(c!.group,g)=5);; total_time_1:=total_time
gap> inv:=Set(l,g->Homology(TensorWithIntegersOverSubgroup(c,g),2)); total_time_1:=total_time_1+t
[ [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
 ]
gap> total_time_1;
3407
gap> hopf:=PureComplexComplement(HopfSatohSurface());;
gap> hopf:=RegularCWComplex(hopf);;
gap> Size(hopf);
4508573
gap> total_time_2:=0;;
gap> c_:=ChainComplexOfUniversalCover(hopf);; total_time_2:=total_time_2+time;;
gap> l_:=Filtered(LowIndexSubgroups(c_!.group,5),g->Index(c_!.group,g)=5);; total_time_2:=total_t
gap> inv_:=Set(l_,g->Homology(TensorWithIntegersOverSubgroup(c_,g),2));; total_time_2:=total_time
gap> total_time_2;
1116000
gap> inv_=inv;
true
```

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