Symbolic algebra and Mathematics with Xcas

Renée De Graeve, Bernard Parisse
University of Grenoble I
Corrected by Jay Belanger
Optimization section written by Luka Marohnić

renee.degraeve@wanadoo.fr
bernard.parisse@ujf-grenoble.fr

Contents

Chapter 1

Index

Index

Chapter 2

Introduction

2.1 Notations used in this manual

In this manual, the information that you enter will be typeset in typewriter font. User input typically takes one of three forms:

- Commands that you enter on the command line.
  For example, to compute the sin of $\pi/4$, you can type

  ```
  sin(pi/4)
  ```

- Commands requiring a prefix key.
  These will be indicated by separating the prefix key and the standard key with a plus `+`. For example, to exit an `Xcas` session, you can type

  ```
  Ctrl+Q
  ```

- Menu commands.
  When denoting menu items, submenus will be connected using ▶. For example, from within `Xcas`, you can choose the `File` menu, then choose the `Open` submenu, and then choose the `File` item. This will be indicated by

  ```
  File ▶ Open ▶ File
  ```

2.2 Interfaces for the `giac` library

The `giac` library is a C++ mathematics library. It comes with two interfaces for users to use it directly; a graphical interface and a command-line interface.

The graphical interface is called `Xcas`, and is the most full-featured interface. As well being able to do symbolic and numeric calculations, it has its own programming language, it can draw graphs, it has a built-in spreadsheet, it can do dynamic geometry and turtle graphics.

The command-line interface can be run inside a terminal. It can also do symbolic and numeric calculations and works with the programming language. In a graphical environment, the command-line interface can also be used to draw graphs.

There is also a web version, which can be run through a browser, either over the internet or from local files. Other programs (for example, TeXmacs) have interfaces for the command-line version.

2.2.1 The Xcas interface

To run Xcas in a graphical environment, it depends on which operating system you are using.

- If you are using Unix, you can usually find an entry for the program in a menu provided by the environment. Otherwise, you can start it from a terminal by typing

  ```
  xcas &
  ```

  If for some reason Xcas becomes unresponsive, you can open a terminal and type

  ```
  killall xcas
  ```

  That will kill any running Xcas processes. When you restart Xcas, you will be asked if you want to resume where you left off using an automatic backup file.

- If you are running Windows, you can use the explorer to go to the directory where Xcas is installed. In that directory will be a file called xcas.bat. Clicking on that file will start Xcas.

- If you are running Mac OS, you can use the Finder to go to the xcas_image.dmg file and double-click it. Then double-click the Xcas disk icon. Finally, to launch Xcas, double-click the Xcas program.

When you start Xcas, a window will pop up with menu entries across the top, a bar indicating information about the current Xcas configuration, and an entry line you can use to enter commands. This interface will be described in more detail later, and you can get help from within Xcas with the menu item

Help▶Interface

2.2.2 The command-line interface

In Unix and MacOS you can run giac from a terminal with the command icas (the command giac also works). There are two ways to use the command-line interface.

If you just want to evaluate one expression, you can give icas the expression (in quotes) as a command line argument. For example, to factor the polynomial $x^2 - 1$, you can type

```
icas 'factor(x^2-1)'
```

at a command prompt. The result will be

```
(x-1)*(x+1)
```

and you will be returned to the operating system command line.

If you want to evaluate several commands, you can enter an interactive giac session by entering the command icas (or giac) by itself at a command prompt. You will then be given a prompt specifically for giac commands, which will look like

```
0>>
```

You can enter a giac command at this prompt and get the result.

```
0>> factor(x^2-1)
(x-1)*(x+1)
1>>
```

After the result, you will be given another prompt for giac commands. You can exit this interactive session by typing Ctrl+D.

You can also run icas in batch mode; that is, you can have icas run giac commands stored in a file. This can be done in Windows as well as Unix and Mac OS. To do this, simply enter

```
icas filename
```

at a command prompt, where *filename* is the name of the file containing the giac commands.

2.2.3 The Firefox interface

You can run giac without installing it by using a javascript-enabled web browser. Using Firefox for this is highly recommended; Firefox will run giac several times faster than Chrome, for example, and Firefox also supports MathML natively.

To run giac through Firefox, you can open the url https://www-fourier.ujf-grenoble.fr/~parisse/giac/xcasen.html. At the top of this page is a button which will open a quick tutorial; the tutorial will also tell you how to install the necessary files to run giac through Firefox without being connected to the internet.

2.2.4 The TeXmacs interface

TeXmacs (http://www.texmacs.org) is a sophisticated word processor with special mathematical features. As well as being designed to nicely typeset mathematics, it can be used as a frontend for various mathematics programs, such as giac.

Once you've started TeXmacs, you can interactively run giac within TeXmacs with the menu command Insert▶Session▶Giac. Once started, you can enter giac commands as you would in the command-line interface. The TeXmacs interface will also have a menu specifically for giac commands.

Within TeXmacs, you can combine giac commands and output with ordinary text. To enter normal text within a giac session, use the menu item Focus▶Insert Text Field Above. You can reenter a giac entry line by clicking on it with a mouse.

Chapter 3

The Xcas interface

3.1 The entry levels

The Xcas interface can run several independent calculation sessions, each session will be contained in a separate tab. Before you understand the Xcas interface, it would help to be familiar with the components of a session.

Each session can have any number of input levels. Each input level will have a number to the left of it; the number is used to identify the input level. Each level can have one of the following:

- A command line.
  This is the default; you can open a new command line with Alt+N.
  You can enter a giac command (or a series of commands separated by semicolons) on a command line and send it to be evaluated by hitting enter. You can also scroll through the command history with Shift+Up and Shift+Down.

  If the output is a number or an expression, then it will appear in blue text in a small area below the input region; this area is an expression editor. There will be a scrollbar and a small M to the right of this area; the M is a menu which gives you various options.

  If the output is a graphic, then it will appear in a graphing area below the input region. To the right of the graphic will be a control panel allowing you to manipulate the graphic.

- An expression editor.
  You can open an expression editor with Alt+E.

- A two-dimensional geometry screen.
  You can open up such a screen with Alt+G. This level will have a screen, as well as a control panel, menus and a command line to control the screen.

- A three-dimensional geometry screen.
  You can open up such a screen with Alt+H. This level will have a screen, as well as a control panel, menus and a command line to control the screen.

- A turtle graphics screen.
  You can open up such a screen with Alt+D. This level will have a screen, as well as a program editor and command line.

- A spreadsheet.
  You can open up a spreadsheet with Alt+T. A spreadsheet will be able to open a graphic screen.

- A program editor.
  You can open up a program editor with Alt+P.

- A comment line. You can open up a comment line with Alt+C.

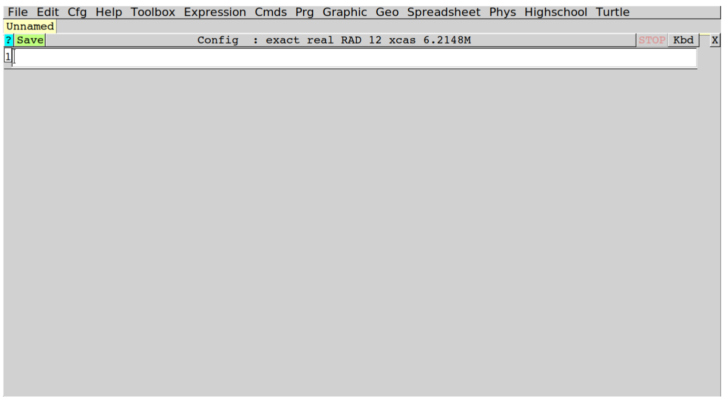
Using commands discussed later, different types of levels can be combined to form a single hybrid level. Levels can also be moved up or down in a session, or even moved to a different session.

The level containing the cursor is the *current level*. The current level can be evaluated or re-evaluated by typing enter.

A level can be selected (for later operations) by clicking on the number in the white box to the left of the level. Once selected, the box containing the number will turn black. You can select a range of levels by clicking on the number for the beginning level, and then holding the shift key while you click on the number for the ending level.

3.2 The starting window

When you first start Xcas, you will be given a largely blank window.



The first row will be the main menus; you can save and load Xcas sessions, configure Xcas and its interface and run various commands with entries from these menus.

The second row will be tabs; one tab for each session that you are running in Xcas. The tabs will contain the name of the sessions, or Unnamed if a session has no name. The first time you start Xcas, there will be only one unnamed session.

The third row will contain various buttons.

- The first button, ?, will open the help index. (The same as the Help▶Index menu entry.) If there is a command on the command line, the help index (see help index ,p.36) will open at this command.

- The second button Save, will save the session in a file. The first time you click on it, you will be prompted for a file name ending in .xws to save the session in. The button will be pink if the session is not saved or if it has changed since the last change, it will be green once the session is saved. The name in the title will be the name of the file used to save the session.

- The third button, which in the picture above is `Config :  exact real RAD 12 xcas 6.2148M`, is a status line indicating the current `Xcas` configuration. (See section 3.5.) If the session is unsaved, it will begin with `Config :`; if the session is saved in a file *filename.xws*, this button will begin with `Config filename.xws :`. Other information on this status line:

  – `exact` or `approx`. (See subsection 3.5.4.) This tells you whether `Xcas` will give you exact values, such as $\sqrt{2}$, when possible or to give you decimal approximations.

  – `real`, `cplx` or `CPLX`. (See subsections 3.5.5 and 3.5.6.) When this shows `real`, then (for example) `Xcas` will by default only find real solutions of equations. When this shows `cplx`, the `Xcas` will find complex solutions of equations. When this shows `CPLX`, then `Xcas` will regard variables as complex; for example, it won't simplify `re(z)` (the real part of the variable z) to `z`.

  – `RAD` or `DEG`. (See subsection 3.5.3.) This tells you whether angles, as in trigonometric arguments, are measured in radians or degrees.

  – An integer. (See subsection 3.5.1, indicating how many significant digits will be used in floating point calculations.

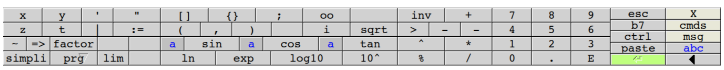
  – `xcas`, `maple`, `mupad` or `ti89`. (See subsection 3.5.2.) This tells you what syntax `Xcas` will use. `Xcas` can be set to emulate the languages of Maple, MuPAD or the TI89 series of calculators.

  – The last item indicates how much memory `Xcas` is using.

  Clicking on this status line button will open a window where you can configure the settings shown on this line as well as some other settings; you can do the same with the menu item `Cfg▶CAS Configuration`. (See subsection 3.5.7.)

- The fourth button, `STOP` (in red), can be used to halt a computation which is running on too long.

- The fifth button, `Kbd`, can be used to toggle an on-screen scientific keyboard at the bottom of the window.



Along the right hand side of the keyboard are some keys that can be used to change the keyboard.

  – The `X` key will hide the keyboard, just like pressing the `Kbd` button again.

  - The `cmds` key will toggle a menu bar at the bottom of the screen which can be used as an alternate menu or persistent submenu. This bar will contain buttons, `home`, `<<`, some menu titles, `>>`, `var`, `cust` and `X`.

    The `<<` and `>>` buttons will scroll through menu items. Clicking on one of the menu buttons will perform the appropriate action or replace the menu items by submenu items. When submenu items appear, there will also be a `BACK` button to return to the previous menu. Clicking on the `home` button returns the menu buttons to the main menu.

    After the menu buttons is a `var` button. This will replace the menu buttons by buttons representing the variables that you have defined. After that is a `cust` button, which will display commands that you store in a list variable `CST`.

    The last button, `X`, will close the menu bar.

  - The `msg` key will bring up a message window at the bottom of the window which will give you helpful messages; for example, if you save a graphic, it will tell you the name of the file it is saved in and how to include it in a LaTeX file.

  - The `abc` key will toggle the keyboard between the scientific keyboard and an alphabetic keyboard.

- The fifth button, `X`, will close the current session.

3.3 Getting help

`Xcas` is an extensive program, but you can get help in several different ways.

Tooltips

If you hover the mouse cursor over certain parts of the `Xcas` window, a temporary window will appear with information about the part. For example, if you move the mouse cursor over the status line, you will get a message saying `Current CAS status. Click to modify.`

If you type a function name into the `Xcas` command line, a similar temporary window will appear with information about the function.

HTML help

If you hit the `F12` button, you will be given a window in which you can use to search the html version of the manual. If you type a string in the search area, you will be given a list of help topics that contain the string. If you choose a topic and click `View`, your web browser will show the appropriate page of the manual.

You can also get HTML help with the menu entry `Help`▶`Find word in HTML help`.

The help index

If you click on the `?` button on the status line you will get the help index.

The help index is a list of the giac function and variable names. Along with the list, the help index window has an area listing words related to any chosen word and words synonymous to the chosen word.

You can scroll through the help index items and click on the word that you want. There is also a line in the help index window that you can use to search the index; you can enter some text and be taken to the part of the index with words beginning with that text. The ? button next to this search line will open the HTML help window.

Below the search line, there is an area which will have a description of the chosen command, and below that is an area which will have examples of the command being used. If the command is a function, then between the description and examples will be some boxes in which you can enter arguments for the command. Filling in these boxes and hitting enter will put the function on the command line.

At the top of the help index window is a Details button. If you click on that, a web page will open up in your browser with the relevant portion of the manual. If you click on the ? next to the search line, you will be taken to the HTML help window.

Besides clicking on the ? on the status line, there are other ways to get to the help index.

- You can get to the help index by using the menu item Help▶Index.
- You can press the tab button while at the Xcas command line to get to the help index. If you have entered part of a command name, you will be at the part of the index with words beginning with the text that you entered.
- If you select a command from the menu, then as well as putting the command on the command line, you will be taken to the help index window with the command chosen.

findhelp

You can get help from Xcas by using the findhelp function. If you enter findhelp(*function*) (or equivalently ?*function*) at the command input, where *function* is the name of a giac function, then some notes on function will appear in the answer portion and the appropriate page of the manual will appear in your web browser.

3.4 The menus

3.4.1 The File menu

The File menu contains commands that are used to save sessions and parts of sessions and load previously saved sessions. This menu contains the following entries:

- New Session
  This will create and open a new session. This session will be in a new tab labeled Unnamed until you save it (using the menu item File▶Save or the keystroke Alt+S).

- `Open`
  This will open a previously saved session. There will be a submenu with a list of saved session files in the primary directory that you can open, as well as a `File` item which will open a directory browser you can use to find a session file. This directory browser can also be opened with `Alt-O`.

- `Import`
  This will allow you to open a session that was created with the Maple CAS, a TI89 calculator or a Voyage200 calculator. These sessions can then be executed with the `Edit`▶`Execute Session` menu entry, but it may be better to execute the commands one at a time to see if any modifications need to be done.

- `Clone`
  This will create a copy of the current session in a Firefox interface; either using the server at http://www-fourier.ujf-grenoble.fr/~parisse/xcasen.html (`Online`) or a local copy (`Offline`).

- `Insert`
  This allows you to insert a previously saved session, a link to a Firefox session, or a previously saved figure, spreadsheet or program.

- `Save` (`Alt+S`)
  This will save the current session.

- `Save as`
  This will save the current session under a different name.

- `Save all`
  This will save all of the sessions.

- `Export as`
  This will allow you to save the current session in different formats; either standard `Xcas` format, Maple format, MuPAD format or TI89 format.

- `Kill`
  This will kill the current session.

- `Print`
  This will allow you to save the session in various ways. `preview` will save an image of the current session in a file that you name. `print` will send an image of the current session to the printer. `preview selected levels` will save the images of the commands and outputs of the current session, each in a separate file.

- `LaTeX`
  This will render the session in LaTeX and give you the result in various ways. `latex preview` will display a compiled LaTeX version of the current session. `latex print` will send a copy of the LaTeXed session to a printer. `latex print selection` will save a copy.

- `Screen capture`
  This will create a screenshot that will be saved in various formats.
- `Quit and update Xcas`
  This will quit `Xcas` after checking for a newer version.
- `Quit` (`Ctrl+Q`)
  This will quit `Xcas`.

3.4.2 The `Edit` menu

The `Edit` menu contains commands that are used to execute and undo parts of the current session. This menu contains the following entries:

- `Execute worksheet` (`Ctrl-F9`)
  This will recalculate each level in the session.
- `Execute worksheet with pauses`
  This will recalculate each level in the session, pausing between calculations.
- `Execute below`
  This will recalculate the current level and each level below it.
- `Remove answers below`
  This will remove the answers to the current level and the levels below it.
- `Undo` (`Ctrl+Z`)
  This will undo the latest edit done to the levels, including the deletion of levels. It can be repeated to undo more than one edit.
- `Redo` (`Ctrl+Y`)
  This will redo the undone editing.
- `Paste`
  This will paste the contents of the system clipboard to the cursor position.
- `Del selected levels`
  This will delete any entry levels that you have selected.
- `selection -> LaTeX` (`Ctrl+T`)
  If you select a level, part of a level, or answer with the mouse (click and drag), this menu item will put a LaTeX version of the selection on the system clipboard.
- `New entry` (`Alt+N`)
  This will insert a new entry level above the current one.
- `New parameter` (`Ctrl+P`)
  This will bring up a window in which you can enter a name and conditions for a new parameter.
- `Insert newline`. This will insert a newline below the cursor. Note that simply typing return will cause the current entry to be evaluated rather than inserting a newline.

- Merge selected levels. This will merge the selected levels into a single level.

3.4.3 The Cfg menu

The Cfg menu contains commands that are used to set the behaviour of Xcas. This menu contains the following entries:

- Cas configuration
  This will open a window that you can use to configure how Xcas performs calculations. This is the same window you get when you click on the status line.

- Graph configuration
  This will open a window that you can use to configure the default settings for a graph. This includes such things as the initial ranges of the variables. Each graph will also have a cfg button to configure the settings on a per graph basis.

- General configuration
  This will open a window that you can use to configure various non-computational aspects of Xcas, such as the fonts, the default paper size, and the like.

- Mode (syntax)
  This will allow you to change the default syntax. To begin with, it is Xcas syntax, but you can change it to Maple syntax, MuPAD syntax or TI89 syntax.

- Show
  This will allow you to control parts of Xcas to show.

  - DispG
    This will show the graphics display screen. This screen will show all graphical commands from the session together.
  - keyboard
    This will show the on-screen keyboard; the same as clicking on the Kbd button on the status line.
  - bandeau
    This will show the menu buttons at the bottom of the window; the same as clicking on cmds on the on-screen keyboard.
  - msg
    This will show the messages window; the same as clicking on msg on the on-screen keyboard.

- Hide
  This will hide the same items that you can show with Show.

- Index language
  This will let you choose a language in which to display the help index.

- `Colors`
  This will let you choose colors for various parts of the display.

- `Session font`
  This will let you choose a font for the sessions.

- `All fonts`
  This will let you choose a font for the session, the main menu and the keyboard.

- `browser`
  This will let you choose a browser that `Xcas` will use when needed. If this is blank, then `Xcas` will use its own internal browser.

- `Save configuration`
  This will save the configurations that you chose with the `Cfg` menu or by clicking on the status line.

3.4.4 The **`Help`** menu

The `Help` menu contains commands that let you get information about `Xcas` from various sources. This menu contains the following entries:

- `Index`
  This will bring up the help index. (See help index , p.36)

- `Find word in HTML help(F12)`
  This will bring up a page which will help you search for keywords in the html documentation that came with `Xcas`. The help will be displayed in your browser.

- `Interface`
  This will bring up a tutorial for the `Xcas` interface. The tutorial will be displayed in your browser.

- `Reference card, fiches`
  This will bring up (in your browser) a pdf reference card for `Xcas`.

- `Manuals`
  This will let you choose from a variety of manuals for XCAS. They will appear in your browser unless otherwise noted.

  – `CAS reference`
    This will bring up a manual for `Xcas`.

  – `Algorithmes (HTML)`
    This will bring up a manual for the algorithms used by `Xcas`.

  – `Algorithmes (PDF)`
    This will bring up a pdf version of the manual for the algorithms used by `Xcas`.

  – `Geometry`
    This will bring up a manual for two-dimensional geometry in `Xcas`.

  - `Programmation`
    This will bring up a manual for programming in `Xcas`.
  - `Simulation`
    This will bring up a manual for statistics and using the `Xcas` spreadsheet.
  - `Turtle`
    This will bring up a manual for using the `Turtle` drawing screen in `Xcas`.
  - `Exercices`
    This will bring up a page of exercises that you can do with `Xcas`.
  - `Amusement`
    This will bring up a page of mathematical amusements that you can work through with `Xcas`.
  - `PARI-GP`
    This will bring up documentation for the GP/PARI functions.

- `Internet`
  The `Internet` menu contains commands that take you to various web pages related to `Xcas`. Among them are the following entries:
  - `Forum`
    This will take you to the `Xcas` forum.
  - `Update help`
    This will install updated help files (retrieved from the `Xcas` website).
- `Start with CAS`
  This menu has the following entries.
  - `Tutorial`
    This opens up the tutorial.
  - `solutions`
    This opens up the solutions to the exercises in the tutorial.
- `Tutoriel algo`
  This opens up a tutorial on algorithms and programming with `Xcas`.
- `Rebuild help cache`
  This will rebuild the help index.
- `About`
  This will display a message window with information about `Xcas`.
- `Examples`
  This will allow you to choose from a variety of example worksheets, which will then be copied to your current directory and opened.

3.4.5 The Toolbox menu

The Toolbox menu contains commands that are used to insert operators into the session. This menu includes the following entries:

- New entry (Alt+N)
  This will insert a new level after the current one.

- New comment (Alt+C)
  This will insert a new comment level after the current level.

The other entries allow you to insert mathematical operations into the current level. When you do that, you will also be taken to the help index (See help index , p.36) with help on the chosen command.

3.4.6 The Expression menu

The Expression menu contains commands that are used to transform expressions. The first entry is New expression (which is equivalent to Alt+E), which will insert a new level above the current level and bring up the on-screen keyboard. The rest of the entries can be used to insert a transformation.

3.4.7 The Cmds menu

The Cmds menu contains various giac functions and constants.

3.4.8 The Prg menu

The Prg menu contains commands that are used to write giac programs. The first entry, Prg▶New program (equivalent to Alt+P) , will insert a program level and bring up the program editor. The other entries are useful commands for writing giac programs.

3.4.9 The Graphic menu

The Graphic menu contains commands that are used to create graphs. The first entry, Graphic▶Attributs (equivalent to Alt+K) , will bring up a window contains different attributes of the graph (such as line width, color, etc.) The other entries are commands for creating and manipulating graphs.

3.4.10 The Geo menu

The Geo menu contains commands that are used to work with two- and three-dimensional geometric figures. The first two entries, Geo▶New figure 2d (equivalent to Alt+G) and Geo▶New figure 3d (equivalent to Alt+H) will create a level for creating two- and three-dimensional figures, respecitively. The other menu items are for working with the figures.

3.4.11 The Spreadsheet menu

The Spreadsheet menu contains commands that are used to work with spreadsheets. The first menu item, Spreadsheet▶New spreadsheet (equivalent to Alt+T), will bring up a window where you can set the size and other attributes of a spreadsheet and then one will be created. The submenus contain commands for working with spreadsheets. Notice that the spreadsheet itself will have menus that are the same as these submenus.

3.4.12 The Phys menu

The Phys menu contains submenus with various categories of constants, as well as functions for converting units.

3.4.13 The Highschool menu

The Highschool menu contains computer algebra commands that are useful at different levels of highschool. There is also a Program submenu with some program control functions.

3.4.14 The Turtle menu

The Turtle menu contains the commands that are used to in a Turtle screen. The first menu item, Turtle▶New turtle, will create a Turtle drawing screen, the other menu items contain commands for working with the screen.

3.5 Configuring Xcas

3.5.1 The number of significant digits

By default Xcas uses and displays 12 significant digits, but you can set the number of digits to other positive integers. If you set the number of significant digits to a number less than 14, then Xcas will use the computer's floating point hardware, and so calculations will be done to more significant digits than you asked for, but only the number of digits that you asked for will be displayed. If you set the number of significant digits to 14 or higher, then both the computations and the display will use that number of digits.

You can set the number of significant digits for Xcas by using the CAS configuration screen (see subsection 3.5.7). The number of significant digits is stored in the variable DIGITS or Digits, so you can also set it by giving the variable DIGITS a new value, as in DIGITS:= 20. The value will be stored in the configuration file (see subsection 3.5.10), and so can also be set there.

3.5.2 The language mode

Xcas has its own language which it uses by default, but you can have it use the language used by Maple, MuPAD or the TI89 calculator.

You can set which language Xcas uses in the CAS configuration screen (see subsection 3.5.7). You can also use the function maple_mode. If you give it an argument of 0, maple_mode(0), then Xcas will use its own language. If you give it an argument of 1, maple_mode(1), then Xcas will use the Maple language. If you give it an argument of 2, maple_mode(2), then Xcas will use the MuPAD language. Finally, if you give it an argument of 3, maple_mode(3), then Xcas will use the TI89 language.

The language you want to use will be stored in the configuration file (see subsection 3.5.10), and so can also be set there.

3.5.3 The units for angles

By default, Xcas will assume that any angles you give (for example, as the argument to a trigonometric function) is being measured in radians. If you want, you can have Xcas use degrees.

You can set which angle measure Xcas uses in the CAS configuration screen (see subsection 3.5.7). Your choice will be stored in the variable angle_radian; this will be 1 if you measure your angles in radians and 0 if you measure your angles in degrees. You can also change which angle measure you use by setting the variable angle_radian to the appropriate value. The angle measure you want to use will be stored in the configuration file (see subsection 3.5.10), and so can also be set there.

3.5.4 Exact or approximate values

Some number, such as π and $\sqrt{2}$, can't be written down exactly as a decimal number. When computing with such numbers, Xcas will leave them in exact, symbolic form. If you want, you can have Xcas automatically give you decimal approximations for these numbers.

You can set whether or not Xcas will give you exact or approximate values from the CAS configuration screen. Your choice will be stored in the variable approx_mode, where a value of 0 means that Xcas should give you exact answers when possible and a value of 1 means that Xcas should give you decimal approximations. Your choice will be stored in the configuration file (see subsection 3.5.10), and so can also be set there.

3.5.5 Complex numbers

When factoring polynomials, Xcas won't introduce complex numbers if they aren't already being used. For example,

```
factor(x^2 + 2)
```

will simply return

```
x^2 + 2
```

but if an expression already involves complex numbers then Xcas will use them;

```
factor(i*x^2 + 2*i)
```

will return

```
(x - i*sqrt(2))*(i*x - sqrt(2))
```

Xcas also has ways of finding complex roots even when complex numbers are not present; for example, the command cfactor will factor over the complex numbers

```
cfactor(x^2 + 2)
```

will return

```
(x - i*sqrt(2))*(x + i*sqrt(2))
```

If you want Xcas to use complex numbers by default, you can turn on complex mode. In complex mode,

```
factor(x^2 + 2)
```

will return

```
(x - i*sqrt(2))*(x + i*sqrt(2))
```

You can turn on complex mode from the CAS configuration screen. This mode is determined by the value of complex_mode; if this is 1 then complex mode is on, if this variable is 0 then complex mode is off. This option will be stored in the configuration file (see subsection 3.5.10), and so can also be set there.

3.5.6 Complex variables

New variables will be assumed to be real; functions which work with the real and imaginary parts of variables will assume that a variable is real. For example, re returns the real part of its argument and im returns the imaginary part, and so

```
re(z)
```

returns

```
z
```

and

```
im(z)
```

returns

```
0
```

If you want variables to be complex by default, you can have Xcas use complex variable mode. You can set this from the CAS configuration screen. Your choice will be stored in the variable complex_variables, where a value of 0 means that Xcas will assume that variables are real and and a value of 1 means that Xcas will assume that values are complex. Your choice will be stored in the configuration file (see subsection 3.5.10), and so can also be set there.

3.5.7 Configuring the computations

You can configure how Xcas computes by using the menu item Cfg▶Cas configuration or by clicking on the status line. You will then be given a window in which you can change the following options:

- Prog style (default: Xcas)
  You will have a menu from which you can choose a different language to program in; you can choose from Xcas, Xcas (Python), Maple, Mupad and TI89/92.
- eval (default: 25)
  You can type in a positive integer indicating the maximum number of recursions allowed when evaluating expressions.
- prog (default: 1)
  You can type in a positive integer indicating the maximum number of recursions allowed when executing programs.
- recurs (default: 100)
  You can type in a positive integer indicating the maximum number of recursive calls.
- debug (default: 0)
  You can type in an integer, 0 or 1. If this is 1, then Xcas will display intermediate information on the algorithms used by giac. If this number is 0, then no such information is displayed.
- maxiter (default: 20)
  You can type in an integer indicating the maximum number of iterations in Newton's method.
- Float format (default: standard)
  You will have a menu from which you can choose how to display decimal numbers. Your choices will be:
  - standard In standard notation, a number will be written out completely without using exponentials; for example, 15000.12 will be displayed as 15000.12.
  - scientific In scientific notation, a number will be written as a number between 1 and 10 times a power of ten; for example, 15000.12 will be displayed as 1.500012000000e+04 (where the number after e indicates the power of 10).
  - engineer In engineer notation, a number will be written as a number between 1 and 1000 times a power of ten, where the power of 10 is a multiple of three. For example, 15000.12 will be displayed as 15.00012e3.
- Digits (default: 12)
  You can enter a positive integer which will indicate the number of significant digits.

- `epsilon` (default: 1e-12)
  You can enter a floating point number which will be the value of epsilon used by `epsilon2zero`, which is a function which replaces numbers with absolute value less than epsilon by 0.

- `proba` (default: 1e-15)
  You can enter a floating point number. If this number is greater than zero, then in some cases `giac` can use probabilistic algorithms and give a result with probability of being false less than this value. (One such example of a probabilistic algorithm that `giac` can use is the algorithm to compute the determinant of a large matrix with integer coefficients.)

- `approx` (default: unchecked)
  You will be given a checkbox. If the box is checked, then exact numbers such as $\sqrt{2}$ will be given a floating point approximation. If the box in unchecked, then exact values will be used when possible.

- `autosimplify` (default: 1)
  You can enter a simplification level of 0, 1 or 2. A value of 0 means no automatic simplification will be done, a value of 1 means grouped simplification will be automatic. A value of 2 means that all simplification will be automatic.

- `threads` (default: 1)
  You can enter a positive integer to indicate the number of threads (for a possible future threaded version).

- `Integer basis` (default: 10)
  You will be given a menu from which you can choose an integer base to work in; your choices will be 8, 10 and 16.

- `radian` (default: `checked`)
  You will be given a checkbox. If the box is checked, then angles will be measured in radians, otherwise they will be measured in degrees.

- `Complex` (default: `unchecked`)
  You will be given a checkbox. If this box is checked, then `giac` will work in complex mode, meaning, for example, that polynomials will be factored with complex numbers if necessary.

- `Cmplx_var` (default: `unchecked`)
  You will be given a checkbox. If this box is checked, then variables will by default be assumed to be complex. For example, the expression `re(z)` won't be simplified to simply `z`. If this box is unchecked, then `re(z)` will be simplified to `z`.

- `increasing power` (default: `unchecked`)
  You will be given a checkbox. If this box is checked, then polynomials will be written out in increasing powers of the variable; otherwise they will be written in decreasing powers.

- All_trig_sol (default: unchecked)
  You will be given a checkbox. If this box is unchecked, then only the primary solutions of trigonometric equations will be given. For example, the solutions of cos(x)=0 will be the pair [-pi/2,pi/2]. If this box is checked, then the solutions of cos(x)=0 will be [(2*n_0*pi + pi)/2], where n_0 can be any integer.

- Sqrt (default: checked)
  You will be given a checkbox. If this box is checked, then the factor command will factor second degree polynomials, even when the roots are not in the field determined by the coefficients. For example, factor(x^2 - 3) will return (x - sqrt(3))*(x + sqrt(3)). If this box is unchecked, then factor(x^2 - 3) will return x^2 - 3.

This page will also have buttons for applying the settings, saving the settings for future sessions, canceling any new settings, or restoring the default settings.

3.5.8 Configuring the graphics

You can configure each graphics screen by clicking on the cfg button on the graphics screen's control panel to the right of the graph. You can also change the default graphical configuration using the the menu item Cfg▶Graph configuration. You will then be given a window in which you can change the following options:

- X- and X+
  These will determine the x values for which calculations will be done.

- Y- and Y+
  These will determine the y values for which calculations will be done.

- Z- and Z+
  These will determine the z values for which calculations will be done.

- t- and t+
  These will determine the t values for which calculations will be done, when plotting parametric curves, for example.

- WX- and WX+
  These will determine the range of x values for the viewing window. done.

- WY- and WY+
  These will determine the range of y values for the viewing window.

- class_min
  This will determine the minimum size of a statistics class.

- class_size
  This will determine the default size of a statistics class.

- autoscale
  When checked, the the graphic will be autoscaled.

- `ortho`
  When checked, all axes of the graphic will be scaled equally.
- `>W` and `W>`
  These are convenient shortcuts to copy the `X-`, `X+`, `Y-` and `Y+` values to `WX-`, `WX+`, `WY-` and `WY+`, or the other way around.

This page will also have buttons for applying the settings, saving the settings for future sessions, or canceling any new settings.

3.5.9 More configuration

You can configure other aspects of `Xcas` (besides the computatioonal aspects and graphics) using the the menu item `Cfg`▶`General configuration`. You will then be given a window in which you can change the following options:

- `Font`
  This lets you choose a session font, the same as choosing the menu item `Cfg`▶`Session font`.
- `Level`
  This will determine what type of level should be open when you start a new session.
- `browser`
  This will determine what browser `Xcas` should use when it requires one, for example when displaying help. If this is empty, `Xcas` will use its built-in browser.
- `Auto HTML help`
  If this box is checked, then whenever you choose a function from a menu, a help page for that function will appear in your browser. Regardless of whether this box is checked or not, the help page will also appear in your browser if you type `?`*function* in a command box.
- `Auto index help` If this box is checked, then whenever you choose a function from a menu, the help index page for that function will appear. This is the same page you would get from choosing the function from the help index.
- `Print format`
  This will determine the paper size for printing and saving files. There is also a button you can use to have the printing done in landscape mode; if this button is not checked, the printing will be done in portrait.
- `Disable Tool tips`
  If this is checked, `Xcas` will stop displaying tool tips.
- `rows` and `columns`
  These will determind the default number of rows and columns for the matrix editor and spreadsheet.
- `PS view`
  This determines what program will be used to preview Postscript files.

3.5.10 The configuration file

When you save changes to your configuration, this is stored in a configuration file, which will be .xcasrc in your home directory in Unix and xcas.rc in Windows. This file will have four functions – widget_size, cas_setup, maple_mode and xyztrange – which determine the configuration and which are evaluated when Xcas starts.

3.6 Printing and saving

3.6.1 Saving a session

Each tab above the status line represents a session, the active tab will be yellow. The label of each tab will be the name of the file that the session is saved in; if the session hasn't been saved the tab will read Unnamed.

You can save your current session by clicking on the Save button on the status line. If the session contains unsaved changes the Save button will be red; the button will be green when nothing needs to be saved. The first time that you save a session you will be prompted for a file name; you should choose a name that ends in .xws. Subsequent times that you save a session it will be saved in the same file; to save a session in a different file you can use the menu item File▶Save as.

If you have a session saved in a file and you want to load it in a tab, you can use the menu item File▶Open. From there you can choose a specific file from a list or open a directory browser that you can use to choose a file. The directory browser can also be opened with Alt-O.

3.6.2 Saving a spreadsheet

If you have a spreadsheet in one of the levels, you can save it separately from the rest of the session.

Once a spreadsheet is inserted, it will have menus right next to the level number. If you select the Table▶Save sheet as text menu, you will be prompted for a file name. You should choose a file name that ends in .tab. Once you save the spreadsheet, there will be a button to the right of the menus which you can use to save any changes you make. If you want to save the spreadsheet under a different name, you can use the Table▶Save as alternate filename menu entry. You can also use the Table▶Save as CSV and Table▶Save as mathml menu entries to save the spreadsheet in other formats.

You can use the Table menu to insert previously saved spreadsheets; the menu item Table▶Insert will bring up a directory browser you can use to select a file to enter.

3.6.3 Saving a program

You can open up a level in which to write an Xcas program with the menu item Prg▶New program (which is equivalent to Alt-P). If you select this item, you will be prompted for information to fill out a template for a program and then be left in the program editor.

At the top of the program editor there will be menus and buttons, at the far right will be a `Save` button that you can press to save the program. The first time you save a program, you will be prompted for a file name, you should choose a name ending in `.cxx`. Once a program is saved, the file name will appear to the right of the `Save` button. If you want to save the program under a different name, you can use the `Prog`▶`Save as` item from the program editor menu.

To insert a previously saved program, you can use the `Prog`▶`Load` item from the program editor menu.

3.6.4 Printing a session

You can print a session with the `File`▶`Print`▶`to printer` menu item.

If you prefer to save the printed form as a file, you can use the `File`▶`Print`▶`preview` menu item. You will prompted for a file name to save the printed form in; the file will be a PostScript file, so the name should end in `.ps`.If you only want to save certain levels in printable form, you can use the `File`▶`Print`▶`preview selected levels` menu item; this file will be encapsulated PostScript, so the name should end in `.eps`.

3.7 Translating to other computer languages

`Xcas` can translate a session, or parts of a session, to other computer languages; notably LaTeX and MathML.

3.7.1 Translating an expression to LaTeX

The command `latex` will translate an expression to a LaTeX expression. If you enter `latex(`*`expression`*`)`, then the expression will be evaluated and the result will be given to you in the LaTeX typesetting language. For example, if you enter

```
latex(1+1/2)
```

you will get

```
\frac{3}{2}
```

3.7.2 Translating the entire session to LaTeX

If you want to save your entire document as a complete LaTeX file, you can use the menu item `File`▶`laTeX preview selection`

Chapter 4

Entry in Xcas

4.1 Suppressing output

If you enter a command into Xcas, the result will appear in the output box below the input. If you enter

```
a := 2+2
```

then

```
4
```

will appear in the output box. You can evaluate the input and suppress the output with the nodisp command. If you enter

```
nodisp(a := 2+2)
```

then a will still be set to 4, but the result will not appear in the output box. Instead,

```
Done
```

will appear.

An alternate way of suppressing the output is to end the input with :;, if you enter

```
b := 3+3:;
```

then b will be set to 6 but it won't be displayed.

4.2 Entering comments

You can annotate an Xcas session by adding comments. You can enter a comment on the current line at any time by typing Alt+C. The line will appear in green text and conclude when you type Enter. Comments are not evaluated and so have no output. If you have begun entering a command when you begin a comment, the command line be pushed down so that you can finish it when you complete the comment.

You can open the browser in a comment line by entering the web address beginning with the @ sign. If you enter the comment line

```
The Xcas homepage is at
@www-fourier.ujf-grenoble.fr/~parisse/giac.html
```

then the browser will open to the `Xcas` home page.

To add a comment to a program, rather than a session, you can use the `comment` command, which takes a string as an argument. Alternatively, any part of a program between `//` and the end of the line is a comment. So both

```
bs() := {comment("Hello"); return "Hi there!";}
```

and

```
bs() := { // Hello
return "Hi there!";}
```

are programs with the comment "Hello".

4.3 Editing expressions

You can enter expressions on the command line, but `Xcas` also has a built-in expression editor that you can use to enter expressions in two dimensions, the way they normally look when typeset. When you have an expression in the editor, you can also manipulate subexpressions apart from the entire expression.

4.3.1 Entering expressions in the editor

The expression

$$\frac{x+2}{x^2-4}$$

can be entered on the command line with

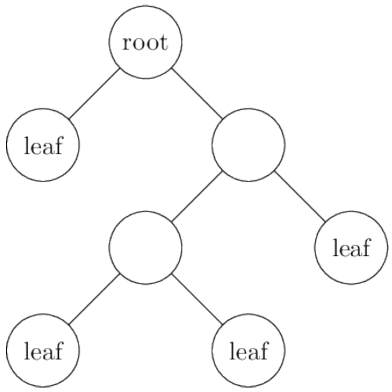
```
(x+2)/(x^2-4)
```

You also can use the expression editor to enter it visually, as $x+2$ on top of x^2-4. To do this, you can start the expression editor with the `Alt+E` keystroke (or the `Expression` ▶ `New Expression` menu command). There will be a small `M` on the right side of the expression line, which is a menu with some commands you can use on the expressions. There will also be a `0` selected on the expression line and an on-screen keyboard at the bottom. If you type `x + 2`, it will overwrite the `0`. To make this the top of the fraction, you can select it with the mouse (you can also make selections with the keyboard, as will be discussed later) and then type `/`. This will leave the `x + 2` on the top and the cursor on the bottom. To enter x^2-4 on the bottom, begin by typing `x`. Selecting this `x` and typing `^2` will put on the superscript. Finally, selecting the $\mathtt{x}^2$ and typing `- 4` will finish the bottom. If you then hit `Enter`, the expression will be evaluated and will appear on the output line.

4.3.2 Subexpressions

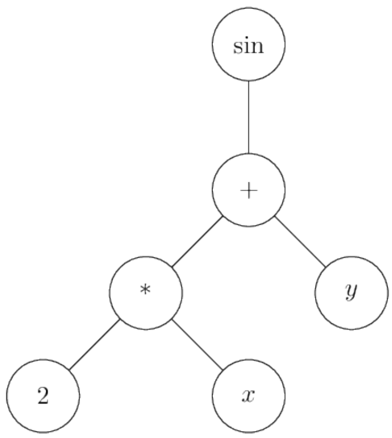
Xcas can operate on expressions in the expression editor or subexpressions of the expression. To understand subexpressions and how to select them, it helps to know that Xcas stores expressions as *trees*.

A tree, in this sense, consists of objects called nodes. A node can be connected to lower nodes, called the children of the first node. Each node (except one) will be connected to exactly one node above it, called the parent node. One special node, called the root node, won't have a parent node. Two nodes with the same parent nodes are called siblings. Finally, if a node doesn't have any children, it is called a leaf. This terminology comes from a visual representation of a tree,



which looks like an upside-down tree; the root is at the top and the leaves are at the bottom.

Given an expression, the nodes of the corresponding tree are the functions, operators, variables and constants. The children of a function node are its arguments, the children of an operator node are its operands, and the constants and variables will be the leaves. For example, the tree for $\sin(2 * x + y)$ will look like

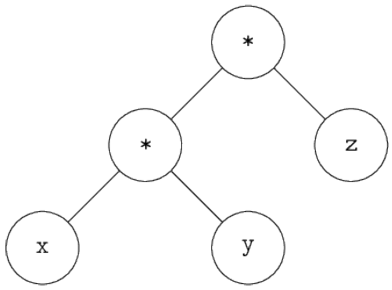


A subexpression of an expression will be a selected node together with the nodes below it. For example, both $2*x$ and $2*x+y$ are subexpressions of $\sin(2*x+y)$, but $x + y$ is not.

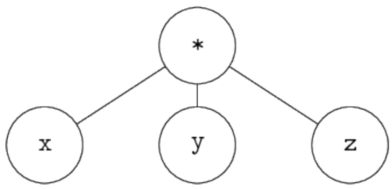
A subexpression of the contents of the expression editor can be selected with the mouse; the selection will appear white on a black background. A subexpression can also be chosen with the keyboard using the arrow keys. Given a selection:

- The up arrow will go to the parent node.
- The down arrow will go to the leftmost child node.
- The right and left arrows will go to the right and left sibling nodes.
- The control key with the right and left arrows will switch the selection with the corresponding sibling.
- If a constant or variable is selected, the backspace key will delete it. For other selections, backspace will delete the function or operator, and another backspace will delete the arguments or operands.

You can use the arrow keys to navigate the tree structure of an expression, which isn't always evident by looking at the expression itself. For example, suppose you enter `x*y*z` in the editor. The two multiplications will be a different levels; the tree will look like



If you select the entire expression with the up arrow and then go to the `M` menu to the right of the line and choose eval, then the expression will look the same but, as you can check by navigating it with the arrow keys, the tree will look like



4.3.3 Manipulating subexpressions

If a subexpression is selected in the expression editor, then any menu command will be applied to that subexpression.

For example, suppose that you enter the expression

```
(x+1)*(x+2)*(x-1)
```

in the expression editor. Note that you can use the abilities of the editor to make this easier. First, enter `x+1`. Select this with the up arrow, then type `*` followed by `x+2`. Select the `x+2` with the up arrow and then type `*` followed by `x-1`. Using the up arrow again will select the `x-1`. Select the entire expression with the up arrow, and then select `eval` from the `M` menu. This will put all factors at the same level. Suppose you want the factors `(x+1)*(x+2)` to be expanded. You could select `(x+1)*(x+2)` with the mouse and do one of the following:

- Select the `Expression▶Misc▶normal` menu item. You will then have `normal((x+1)*(x+2))*(x-1)` in the editor. If you hit enter, the result $(x^2 + 3x + 2) * (x - 1)$ will appear in the output window.

- Again, select the `Expression▶Misc▶normal` menu item, so again you have `normal((x+1)*(x+2))*(x-1)` in the editor. Now if you select `eval` from the `M` menu, then the expression in the editor will become the result $(x^2 + 3x + 2) * (x - 1)$, which you can continue editing.

- Choose `normal` from the `M` menu. This will apply normal to the selection, and again you will have the result $(x^2 + 3x + 2) * (x - 1)$ in the editor.

There are also keystroke commands that you can use to operate on subexpressions that you've selected. There are the usual `Ctrl+Z` and `Ctrl+Y` for undoing and redoing. Some of the others are given in the following table.

Key	**Action on selection**
`Ctrl+D`	differentiate
`Ctrl+F`	factor
`Ctrl+L`	limit
`Ctrl+N`	normalize
`Ctrl+P`	partial fraction
`Ctrl+R`	integrate
`Ctrl+S`	simplify
`Ctrl+T`	copy LaTeX version to clipboard

4.4 Spreadsheet

4.4.1 Opening a spreadsheet

You can open a spreadsheet (or a matrix editor) with the `Spreadsheet▶New Spreadsheet` menu item or with the key `Alt+T`.

When you open a new spreadsheet, you will be given a configuration screen. The configuration screen allows you to set the following options:

- `Variable` The name of the file where the spreadsheet will be saved.

- `Rows` and `Columns` The number of rows and columns in the spreadsheet.

- `Eval` Whether or not to automatically re-evaluate the entries in the spreadsheet after each change. If this is not checked, then you can re-evaluate the spreadsheet with the `eval` button on the spreadsheet menu bar.

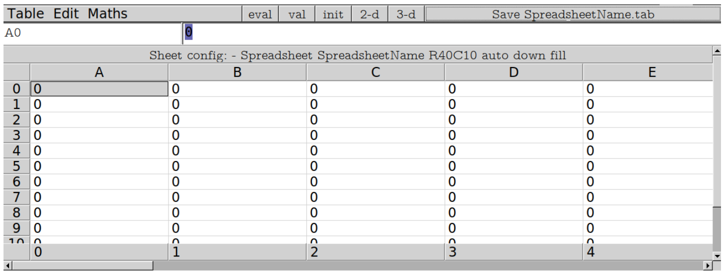
- `Distribute` Whether or not entering a matrix into a cell will keep the entry in a single cell or distribute it across an appropriate array of cells.

- `Landscape` Whether the graphical representation of the spreadsheet should be displayed below the spreadsheet or to the right of the spreadsheet. If this is checked, it will be displayed below the spreadsheet.

- `Move right` Whether or not to move to the cell to the right of the current cell when data is entered. If this is not checked, you will be moved to the cell below the current cell.
- `Spreadsheet` Whether to format a spreadsheet or a matrix.
- `Graph` Whether or not to display the graphical representation of the spreadsheet.

4.4.2 The spreadsheet window

When you open a spreadsheet, the input line will become the spreadsheet.



The top will be a menu bar with `Table`, `Edit` and `Maths` menus as well as `eval`, `val`, `init`, `2-d` and `3-d` buttons. To the right will be the name of the file the spreadsheet will be saved into. Below the menu bar will be two boxes; a box which displays the active cell (and can be used to choose a cell) and a command line to enter information into the cell. Below that will be a status line, you can click on this to return to the configuration screen.

4.5 Variables

4.5.1 Variable names

A variable or function name is a sequence of letters, numbers and underscores that begins with a letter. If you define your own variable or function, you can't use the names of built-in variables or functions, or other keywords reserved by `Xcas`.

4.5.2 Assigning values

You can assign a value to a variable with the := operator. For example, to give the variable `a` the value of `4`, you can enter

```
a := 4
```

Alternatively, you can use the => operator; when you use this operator, the value comes before the variable;

```
4 => a
```

The function `sto` or `Store` can also be used; again, the value comes before the variable

```
sto(4,a)
```

After any one of these commands, any time you use the variable `a` in an expression, it will be replaced by `4`.

4.5.3 Incrementing variables

You can increase the value of a variable a by 4, for example, with

```
a := a + 4
```

If beforehand a were equal to 4, it would now be equal to 8. A shorthand way of doing this is with the += operator;

```
a += 4
```

will also increase the value of a by 4.

Similar shorthands exist for subtraction, multiplication and division. If a is equal to 8 and you enter

```
a -= 2
```

then a will be equal to 6. If you follow this with

```
a *= 3
```

then a will be equal to 18, and finally

```
a /= 9
```

will end with a equal to 2.

4.5.4 Storing and recalling variables and their values

You can store variables and their values for later use in a file of your choosing with the archive function. This function takes two arguments, a filename to store the variables in and a variable or list of variables.

If you have given the variable a the value 2 and the variable bee the value "letter" (a string), then entering

```
archive("foo",[a,bee])
```

will create a file named "foo" which contains the values 2 and "letter" in a format meant to be efficiently read by Xcas.

You can recall the values stored by archive with the unarchive command, which takes a file name as argument. If the file "foo" is as above, then

```
unarchive("foo")
```

will result in

```
[2, letter]
```

If you want to reassign these values to a and bee, you can enter

```
[a,bee] := unarchive("foo")
```

4.5.5 Copying variables

If a variable has a value, such as

```
a := 1
```

and you set a second variable to the first variable

```
b := a
```

the new variable will have the same value as the first; in this case `b` will be equal to `1`. If you later give the first variable a new value;

```
a := 5
```

the new value will still have the old value, in this case, `b` will still be equal to `1`.

The `CopyVar` command will copy one variable to another without evaluating the first variable; the new variable will simply be a copy of the first. With `a` having the value of `5`, as above, the command

```
CopyVar(a,c)
```

will make `c` a copy of the variable `a`, so it will have the value `5` also. If you now change the value of `a`

```
a := 10
```

then the value of `c` will also change; here, `c` will now have the value `10`.

4.5.6 Assumptions on variables

If you enter

```
abs(var)
```

the `Xcas` will return it unevaluated, since `Xcas` doesn't know what type of value the variable is supposed to represent.

The `assume` (or `supposons`) command will let you tell `Xcas` some properties of a variable without giving the variable a specific value. For example, if you enter

```
assume(var > 0)
```

then `Xcas` will assume that `var` is a positive real number, and so for example

```
abs(var)
```

will be evaluated to

```
var
```

You can put one or more conditions in the `assume` command by combining them with `and` and `or`. For example, if you want the variable `a` to be in $[2,4) \cup (6,\infty)$, you can enter

```
assume((a >= 2 and a < 4) or a > 6)
```

If a variable has attached assumptions, then making another assumption with `assume` will remove the original assumptions. To add extra assumptions, you can either use the `additionally` command or give `assume` a second argument of `additionally`. If you assume that $b > 0$ with

```
assume(b > 0)
```

and you want to add the condition that $b < 1$, you can either enter

```
assume(b < 1, additionally)
```

or

```
additionally(b < 1)
```

As well as equalities and inequalities, you can make assumptions about the domain of a variable. If you want `n` to represent a positive integer, for example, you can enter

```
assume(n, integer)
```

If you want `n` to be a positive integer, you can add the condition

```
additionally(n > 0)
```

You can use the `about` command to check the assumptions on a variable; for the above positive integer `n`, if you enter

```
about(n)
```

you will get

```
assume[integer,[line[0,+infinity]],[0]]
```

The first element tells you that `n` is an integer, the second element tells you that `n` is between `0` and `+infinity`, and the third element tells you that the value `0` is excluded.

If you assume that a variable is equal to a specific value, such as

```
assume(c = 2)
```

then by default the variable `c` will remain unevaluated in later levels. If you want an expression involving `c` to be evaluated, you would need to put the expression inside the `evalf` command; if you enter

```
evalf(c^2 + 3)
```

then you will get

```
7.0
```

Right below the `assume(c = 2)` command line there will be a slider, namely arrows pointing left and right with the value `2` between them. These can be used to change the values of `c`. If you click on the right arrow, the `assume(c = 2)` command will transform to

```
assume(c=[2.2,-10.0,10.0,0.0])
```

and the value between the arrows will be `2.2`. Also, any later levels where the variable `c` is evaluated will be re-evaluated with the value of `c` now `2.2`. The output to `evalf(c^2 + 3` will become

```
7.84
```

The `-10.0` and `10.0` in the `assume` line represent the smallest and largest values that `c` can become using the sliders. You can set them yourself in the `assume` command, as well as the increment that the value will change; if you want `c` to start with the value `5` and vary between `2` and `8` in increments of `0.05`, then you can enter

```
assume(c = [5,2,8,0.05])
```

You can remove any assumptions you have made about a variable with the `purge` command; if you enter

```
purge(a)
```

then `a` will no longer have any assumptions made about it. You can remove assumptions from more than one variable at a time;

```
purge(a,b)
```

will remove any assumptions about `a` and `b`.

4.5.7 Unassigning variables

The `VARS()` command will list the variables to which you have assigned values or assumptions. If you begin by entering

```
a := 1
```

and

```
anothervar := 2
```

then

```
VARS()
```

will return

```
[a, anothervar]
```

The `purge` command will clear the values and assumptions you make on variables. You can clear the values and assumptions you have made on all variables with

```
restart
```

or

```
rm_all_vars()
```

command.

4.6 Functions

4.6.1 Defining functions

You can use the := and => operators to define functions; both

```
f(x) := x^2
```

and

```
x^2 => f(x)
```

give the name `f` to the function which takes a value and returns the square of the value. If you then enter

```
f(3)
```

you will get

```
9
```

You can give `Xcas` a function without a name with the `->` operator; the squaring function can be written without a name as

```
x -> x^2
```

You can use this form of the function to assign it to a name; both

```
f := x -> x^2
```

and

```
x -> x^2 => f
```

are alternate ways to define `f` as the squaring function.

You can similarly define functions of more than one variable. For example, to define a function which takes the lengths of the two legs of a right triangle and returns the hypotenuse, you could enter

```
hypot(a,b) := sqrt(a^2 + b^2)
```

or

```
hypot := (a,b) -> sqrt(a^2 + b^2)
```

4.6.2 Defining piecewise defined functions

You can use `Xcas`'s control structures to define functions not given by a single simple formula. Notably, you can use the `ifte` command or `?  :` operator to define piecewise-defined functions.

The `ifte` command takes three arguments; the first argument is a condition, the second argument tells the command what to return when the condition is true, and the third argument tells the command what to return when the condition is false. For example, you could define your own absolute value function with

```
myabs(x) := ifte(x >= 0, x -1*x)
```

Afterwards, for example, entering

```
myabs(-4)
```

will return

```
4
```

However, this will return an error if it can't evaluate the conditional. For example, if you enter

```
myabs(x)
```

you will get the error

```
Ifte: Unable to check test Error: Bad Argument
Value
```

The `?:` construct behaves similarly to `ifte` but is structured differently. Here, the condition comes first, followed by `?`, then what to return if the condition is true, followed by the `:`, and then what to return if the condition is false. You could define your absolute value function with

```
myabs(x) := (x >= 0)? x: -1*x
```

If you enter

```
myabs(-4)
```

you will again get

```
4
```

but now if the conditional can't be evaluated, you won't get an error.

```
myabs(x)
```

will return

```
((x >= 0)? x: -x)
```

The `when` and `IFTE` commands are synonyms for the `?   :` construct;

`(`*condition*`)? ` *true-result*`: ` *false-result*

`when(`*condition*`, ` *true-result*`, ` *false-result*`)`

and

`IFTE(`*condition*`, ` *true-result*`, ` *false-result*`)`

all represent the same expression.

If you want to define a function with several pieces, it may be simpler to use the `piecewise` function. The arguments to this function are alternately conditions and results to return if the condition is true, with the last argument being what to return if none of the conditions are true. For example, to define the function given by

$$f(x) = \begin{cases} -2 & \text{if } x < -2 \\ 3x+4 & \text{if } -2 \le x < -1 \\ 1 & \text{if } -1 \le x < 0 \\ x+1 & \text{if } x \ge 0 \end{cases}$$

you can enter

```
f(x) := piecewise(x < -2, -2, x < -1, 3*x+4, x <
0, 1, x + 1)
```

4.7 Directories

4.7.1 Working directories

Xcas has a working directory that it uses to store files that it creates; typically the user's home directory. You can print the name of the current working directory with the pwd() command; if you enter

```
pwd()
```

you might get something like

```
/home/username
```

You can change the working directory with the cd command; if you enter

```
cd("foo")
```

or (on a Unix system)

```
cd("/home/username/foo")
```

will change to the directory foo, if it exists. Afterwards, any files that you save from Xcas will be in that directory.

If you have values saved in a file, then you'll need to be in that working directory to load it. Note that if you have the same file name in different directories, then the result of loading the file name will depend on which directory you are in.

4.7.2 Reading files

If you have a function or other Xcas information in a file, you can load it with the read function. If the file is named myfunction.cxx, then

```
read("myfunction.cxx")
```

will load the file, as long as the directory is in the current working directory. If the file is in a different directory, you can still load it by giving the path to the file,

```
read("/path/to/file/myfunction.cxx")
```

While read can be used to load files containing Xcas functions, which typically end in .cxx, if you want to load a saved session you should use the load function;

```
load("mysession.cas")
```

4.7.3 Internal directories

You can create a directory that isn't actually on your hard drive but is treated like one from `Xcas`. You can create such an internal directory with the `NewFold` command, which takes a variable name as an argument. If you enter

```
NewFold(MyIntDir)
```

then there will be a new internal directory named `MyIntDir`. Internal directories will also be listed with the `VARS()` command. To actually use this directory, you'll have to use the `SetFold` command;

```
SetFold(MyIntDir)
```

Finally, we can print out the internal directory that we are in with the `GetFold` command; entering

```
GetFold()
```

will result in

```
MyIntDir
```

Afterwards, if this directory is empty, you can delete it with the `DelFold` command;

```
DelFold(MyIntDir)
```

Chapter 5

The CAS functions

5.1 Symbolic constants : `e pi infinity i`

`e` is the number $\exp(1)$;
`pi` is the number π.
`infinity` is unsigned ∞.
`+infinity` is $+\infty$.
`-infinity` is $-\infty$.
`i` is the complex number i.

5.2 Booleans

5.2.1 The values of a boolean : `true false`

The value of a boolean is `true` or `false`.
The synonyms are :
`true` or `TRUE` or `1`,
`false` or `FALSE` or `0`.
Tests or conditions are boolean functions.

5.2.2 Tests : `==, !=, >, >=, <, =<`

`==, !=, >, >=, <, =<` are infixed operators.
`a==b` tests the equality between `a` and `b` and returns `1` if `a` is equal to `b` and `0` otherwise.
`a!=b` returns `1` if `a` and `b` are different and `0` otherwise.
`a>=b` returns `1` if `a` is greater than or equal to `b` and `0` otherwise.
`a>b` returns `1` if `a` is strictly greater than `b` and `0` otherwise.
`a<=b` returns `1` if `a` is less than or equal to `b` and `0` otherwise.
`a<b` returns `1` if `a` is strictly less than `b` and `0` otherwise.
To write an algebraic function having the same result as an `if...then...else`, we use the boolean function `ifte`.
For example :

```
f(x):=ifte(x>0,true,false)
```

defines the boolean function f such that `f(x)= true` if $x \in (0; +\infty[$ and `f(x)=false` if $x \in (-\infty; 0]$.
Input :

```
f(0)==0
```

Output :

```
1
```

Look out !
`a=b` is not a boolean !!!!
`a==b` is a boolean.

5.2.3 Boolean operators : `or xor and not`

`or` (or `||`), `xor`, `and` (or `&&`) are infixed operators.
`not` is a prefixed operators.
If `a` and `b` are two booleans :
`(a or b) (a || b)` returns `0` (or `false`) if `a` and `b` are equal to 0 and returns `1` (or `true`) otherwise.
`(a xor b)` returns `1` if `a` is equal to 1 and `b` is equal to 0 or if `a` is equal to 0 and `b` is equal to 1 and returns 0 if `a` and `b` are equal to 0 or if `a` and `b` are equal to 1 (it is the "exclusive or").
`(a and b)` or `(a && b)` returns `1` (or `true`) if `a` and `b` are equal to 1 and `0` (or `false`) otherwise.
`not(a)` returns `1` (or `true`) if `a` is equal to 0 (or `false`), and `0` (or `false`) if `a` is equal to 1 (or `true`).
Input :

```
1>=0 or 1<0
```

Output :

```
1
```

Input :

```
1>=0 xor 1>0
```

Output :

```
0
```

Input :

```
1>=0 and 1>0
```

Output :

```
1
```

Input :

```
not(0==0)
```

Output :

```
0
```

5.2.4 Transform a boolean expression to a list : `exp2list`

`exp2list` returns the list `[expr0,expr1]` when the argument is `(var=expr0) or (var=expr1)`.
`exp2list` is used in TI mode for easier processing of the answer to a `solve` command.
Input :

```
exp2list((x=2) or (x=0))
```

Output :

```
[2,0]
```

Input :

```
exp2list((x>0) or (x<2))
```

Output :

```
[0,2]
```

In TI mode input :

```
exp2list(solve((x-1)*(x-2)))
```

Output :

```
[1,2]
```

5.2.5 Evaluate booleans : `evalb`

Inside Maple, `evalb` evaluates an boolean expression. Since `Xcas` evaluates booleans automatically, `evalb` is only here for compatibility and is equivalent to `eval`
Input :

```
evalb(sqrt(2)>1.41)
```

or :

```
sqrt(2)>1.41
```

Output :

```
1
```

Input :

```
evalb(sqrt(2)>1.42)
```

or :

```
sqrt(2)>1.42
```

Output :

```
0
```

5.3 Bitwise operators

5.3.1 Operators bitor, bitxor, bitand

The integers may be written using hexadecimal notation 0x... for example 0x1f represents 16+15=31 in decimal. Integers may also be output in hexadecimal notation (click on the red CAS status button and select `Base (Integers)`).
`bitor` is the logical inclusive `or` (bitwise).
Input :

`bitor(0x12,0x38)`

or :

`bitor(18,56)`

Output :

`58`

because :
`18` is written `0x12` in base 16 or `0b010010` in base 2,
`56` is written `0x38` in base 16 or `0b111000` in base 2,
hence `bitor(18,56)` is `0b111010` in base 2 and so is equal to `58`.

`bitxor` is the logical exclusive `or` (bitwise).
Input :

`bitxor(0x12,0x38)`

or :

`bitxor(18,56)`

Output :

`42`

because :
`18` is written `0x12` in base 16 and `0b010010` in base 2,
`56` is written `0x38` in base 16 and `0b111000` in base 2,
`bitxor(18,56)` is written `0b101010` in base 2 and so, is equal to `42`.

`bitand` is the logical `and` (bitwise).
Input :

`bitand(0x12,0x38)`

or :

`bitand(18,56)`

Output :

`16`

because :
`18` is written `0x12` in base 16 and `0b010010` in base 2,
`56` is written `0x38` in base 16 and `0b111000` in base 2,
`bitand(18,56)` is written `0b010000` in base 2 and so is equal to `16`.

5.3.2 Bitwise Hamming distance : `hamdist`

The Hamming distance is the number of differences of the bits of the two arguments.
Input :

`hamdist(0x12,0x38)`

or :

`hamdist(18,56)`

Output :

`3`

because :
`18` is written `0x12` in base 16 and `0b010010` in base 2,
`56` is written `0x38` in base 16 and `0b111000` in base 2,
`hamdist(18,56)` is equal to `1+0+1+0+1+0` and so is equal to `3`.

5.4 Strings

5.4.1 Character and string : `"`

`"` is used to delimit a string. A character is a string of length one.
Do not confuse `"` with `'` (or `quote`) which is used to avoid evaluation of an expression . For example, `"a"` returns a string of one character but `'a'` or `quote(a)` returns the variable `a` unevaluated.

When a string is input in a command line, it is evaluated to itself hence the output is the same string. Use + to concatenate two strings or a string and another object.
Example :
Input :

`"Hello"`

`"Hello"` is the input and also the output.
Input :

`"Hello"+", how are you?"`

Output :

`"Hello, how are you?"`

Index notation is used to get the n-th character of a string, (as for lists). Indices begin at 0 in Xcas mode, 1 in other modes.
Example :
Input :

`"Hello"[1]`

Output :

`"e"`

5.4.2 First character, middle and end of a string : `head mid tail`

- `head(s)` returns the first character of the string `s`.
  Input :

  ```
  head("Hello")
  ```

  Output :

  ```
  "H"
  ```

- `mid(s,p,q)` returns the part of the string `s` of size `q` beginning with the character at index `p`.
  Remember that the first index is 0 in Xcas mode.
  Input :

  ```
  mid("Hello",1,3)
  ```

  Output :

  ```
  "ell"
  ```

- `tail(s)` returns the string `s` without its first character.
  Input :

  ```
  tail("Hello")
  ```

  Output :

  ```
  "ello"
  ```

5.4.3 Concatenation of a sequence of words : `cumSum`

`cumSum` works on strings like it does on expressions by doing partial concatenation.
`cumSum` takes as argument a list of strings.
`cumSum` returns a list of strings where the element of index k is the concatenation of the strings with indices 0 to k .
Input :

```
cumSum("Hello, ","is ","that ","you?")
```

Output :

```
"Hello, ","Hello, is ","Hello, is that ","Hello, is
that you?
```

5.4.4 ASCII code of a character : `ord`

`ord` takes as argument a string `s` (resp. a list `l` of strings).
`ord` returns the ASCII code of the first character of `s` (resp. the list of the ASCII codes of the first character of the elements of `l`).
Input :

```
ord("a")
```

Output :

```
97
```

Input :

```
ord("abcd")
```

Output :

```
97
```

Input :

```
ord(["abcd","cde"])
```

Output :

```
[97,99]
```

Input :

```
ord(["a","b","c","d"])
```

Output :

```
[97,98,99,100]
```

5.4.5 ASCII code of a string : `asc`

`asc` takes as argument a string `s`.
`asc` returns the list of the ASCII codes of the characters of `s`.
Input :

```
asc("abcd")
```

Output :

```
[97,98,99,100]
```

Input :

```
asc("a")
```

Output :

```
[97]
```

5.4.6 String defined by the ASCII codes of its characters : `char`

`char` takes as argument a list `l` of ASCII codes.
`char` returns the string whose characters have as ASCII codes the elements of the list `l`.
Input :

```
char([97,98,99,100])
```

Output :

```
"abcd"
```

Input :

```
char(97)
```

Output :

```
"a"
```

Input :

```
char(353)
```

Output :

```
"a"
```

because:
$353 - 256 = 97$.

5.4.7 Find a character in a string : `inString`

`inString` takes two arguments : a string `S` and a character `c`.
`inString` tests if the character `c` is in the string `S`.
`inString` returns the index of its first occurrence or `-1` if `c` is not in `S`.
Input :

```
inString("abcded","d")
```

Output :

```
3
```

Input :

```
inString("abcd","e")
```

Output :

```
-1
```

5.4.8 Concat objects into a string : `cat`

`cat` takes as argument a sequence of objects.
`cat` concatenates these objects into a string.
Input :

```
cat("abcd",3,"d")
```

Output :

```
"abcd3d"
```

Input :

```
c:=5
```

```
cat("abcd",c,"e")
```

Output :

```
"abcd5e"
```

Input :

```
purge(c)
```

```
cat(15,c,3)
```

Output :

```
"15c3"
```

5.4.9 Add an object to a string : +

+ is an infixed operator (resp. `'+'` is a prefixed operator).
If + (resp. `'+'`) takes as argument a string (resp. a sequence of objects with a string as first or second argument), the result is the concatenation of these objects into a string.
warning
+ is infixed and `'+'` is prefixed.
Input :

```
'+'("abcd",3,"d")
```

Output :

```
"abcd"+3+"d"
```

Output :

```
"abcd3d"
```

Input :

```
c:=5
```

Then input:

```
"abcd"+c+"e"
```

or :

```
'+'("abcd",c,"d")
```

Output :

```
"abcd5e"
```

5.4.10 Transform an integer into a string : `cat +`

Use `cat` with the integer as argument, or add the integer to an empty string
Input :

```
""+123
```

or :

```
cat(123)
```

Output :

```
"123"
```

5.4.11 Transform a string into a number : `expr`

Use `expr`, the parser with a string representing a number.

- For integers, enter the string representing the integer without leading 0 for basis 10, with prefix `0x` for basis 16, `0` for basis 8 or `0b` for basis 2. Input :

  ```
  expr("123")
  ```

  Output :

  ```
  123
  ```

  Input :

  ```
  expr("0123")
  ```

  Output :

  ```
  83
  ```

  because :
  $1 * 8^2 + 2 * 8 + 3 = 83$
  Input :

  ```
  expr("0x12f")
  ```

Output :

```
303
```

Because : $1 * 16^2 + 2 * 16 + 15 = 303$

- For decimal numbers, use a string with a `.` or `e` inside.
  Input :

```
expr("123.4567")
```

Output :

```
123.4567
```

Input :

```
expr("123e-5")
```

Output :

```
0.00123
```

- Note that `expr` more generally transforms a string into a command if the command exists.
  Input :

```
expr("a:=1")
```

Output :

```
1
```

Then, input :

```
a
```

Output :

```
1
```

5.5 Write an integer in base b: convert

convert or convertir can do different kind of conversions depending on the option given as the second argument.

To convert an integer n into the list of its coefficients in base b, the option is base. The arguments of convert or convertir are an integer n, base and b, the value of the basis.
convert or convertir returns the list of coefficients in a b basis of the integer n.
Input :

```
convert(123,base,8)
```

Output :

```
[3,7,1]
```

To check the answer, input expr("0173") or horner(revlist([3,7,1]),8) or convert([3,7,1],base,8), the output is 123
Input :

```
convert(142,base,12)
```

Output :

```
[10,11]
```

To convert the list of coefficients of an integer n in base b, the option is also base. convert or convertir returns the integer n.
Input :

```
convert([3,7,1],base,8)
```

or :

```
horner(revlist([3,7,1]),8)
```

Output :

```
123
```

Input :

```
convert([10,11],base,12)
```

or :

```
horner(revlist([10,11]),12)
```

Output :

```
142
```

5.6 Integers (and Gaussian Integers)

For all functions in this section, you can use Gaussian integers (numbers of the form $a + ib$, where a and b are in $\mathbb{Z}$) in place of integers.

5.6.1 The factorial : `factorial`

`Xcas` can manage integers with unlimited precision, such as the following:
Input :

```
factorial(100)
```

Output :

```
933262154439441526816992388562667004907159682643816214685929638952175999932299156089414639761565182862536979208272237582511852109168640000000000000000000000000
```

5.6.2 GCD : `gcd igcd`

`gcd` or `igcd` denotes the gcd (greatest common divisor) of several integers (for polynomials, see also 5.26.7).
`gcd` or `igcd` returns the `GCD` of integers.
Input :

```
gcd(18,15)
```

Output :

```
3
```

Input :

```
gcd(18,15,21,36)
```

Output :

```
3
```

Input :

```
gcd([18,15,21,36])
```

Output :

```
3
```

We can also put as parameters two lists of same size (or a matrix with 2 rows), in this case `gcd` returns the greatest common divisor of the elements with same index (or in the same column).
Input :

```
gcd([6,10,12],[21,5,8])
```

or :

```
gcd([[6,10,12],[21,5,8]])
```

Output :

```
[3,5,4]
```

An example
Find the greatest common divisor of $4n + 1$ and $5n + 3$ when $n \in \mathbb{N}$.
Input :

```
f(n):=gcd(4*n+1,5*n+3)
```

Then, input :

```
essai(n):={
  local j,a,L;
  L:=NULL;
  for (j:=-n;j<n;j++) {
    a:=f(j);
    if (a!=1) {
      L:=L,[j,a];
    }
  }
  return L;
}
```

Then, input :

```
essai(20)
```

Output :

```
[-16,7],[-9,7],[-2,7],[5,7],[12,7],[19,7]
```

So we now have to prove that :
If $n \neq 5+k*7$ (for $k \in \mathbb{Z}$), $4n+1$ and $5n+3$ are mutually prime, and $n = 5+k*7$ (for $k \in \mathbb{Z}$), then the greatest common divisor of $4n + 1$ and $5n + 3$ is 7.

5.6.3 GCD : `Gcd`

`Gcd` is the inert form of `gcd`. See the section **??** for polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$ for using this instruction.
Input :

```
Gcd(18,15)
```

Output :

```
gcd(18,15)
```

5.6.4 GCD of a list of integers : `lgcd`

`lgcd` has a list of integers (or of a list of polynomials) as argument.
`lgcd` returns the `gcd` of all integers of the list (or the `gcd` of all polynomials of the list).
Input :

```
lgcd([18,15,21,36])
```

Output :

```
3
```

Remark
`lgcd` does not accept two lists (even if they have the same size) as arguments.

5.6.5 The least common multiple : `lcm`

`lcm` returns the least common multiple of two integers (or of two polynomials, see also 5.26.10).
Input :

```
lcm(18,15)
```

Output :

```
90
```

5.6.6 Decomposition into prime factors : `ifactor`

`ifactor` has an integer as parameter.
`ifactor` decomposes an integer into its prime factors.
Input :

```
ifactor(90)
```

Output :

```
2*3^2*5
```

Input :

```
ifactor(-90)
```

Output :

```
(-1)*2*3^2*5
```

5.6.7 List of prime factors : `ifactors`

`ifactors` has an integer (or a list of integers) as parameter.
`ifactors` decomposes the integer (or the integers of the list) into prime factors, but the result is given as a list (or a list of lists) in which each prime factor is followed by its multiplicity.
Input :

```
ifactors(90)
```

Output :

```
[2,1,3,2,5,1]
```

Input :

`ifactors(-90)`

Output :

`[-1,1,2,1,3,2,5,1]`

Input :

`ifactor([36,52])`

Output :

`[[2,2,3,2],[2,2,13,1]]`

5.6.8 Matrix of factors : maple_ifactors

`maple_ifactors` has an integer n (or a list of integers) as parameter.
`maple_ifactors` decomposes the integer (or the integers of the list) into prime factors, but the output follows the Maple syntax :
it is a list with +1 or -1 (for the sign) and a matrix with 2 columns and where the lines are the prime factors and their multiplicity (or a list of lists...).
Input :

`maple_ifactors(90)`

Output :

`[1,[[2,1],[3,2],[5,1]]]`

Input :

`maple_ifactor([36,52])`

Output :

`[[1,[[2,2],[3,2]]],[1,[[2,2],[13,1]]]]`

5.6.9 The divisors of a number : idivis divisors

`idivis` or `divisors` gives the list of the divisors of a number (or of a list of numbers).
Input :

`idivis(36)`

Output :

`[1,2,4,3,6,12,9,18,36]`

Input :

`idivis([36,22])`

Output :

`[[1,2,4,3,6,12,9,18,36],[1,2,11,22]]`

5.6.10 The integer Euclidean quotient : `iquo intDiv`

`iquo` (or `intDiv`) returns the integer quotient q of the Euclidean division of two integers a and b given as arguments. ($a = b * q + r$ with $0 \leq r < b$).
For Gaussian integers, we choose q so that $b * q$ is as near by a as possible and it can be proved that r may be chosen so that $|r|^2 \leq |b|^2/2$.
Input :

```
iquo(148,5)
```

Output :

```
29
```

`iquo` works with integers or with Gaussian integers.
Input :

```
iquo(factorial(148),factorial(145)+2 )
```

Output :

```
3176375
```

Input :

```
iquo(25+12*i,5+7*i)
```

Output :

```
3-2*i
```

Here $a - b * q = -4 + i$ and $| - 4 + i|^2 = 17 < |5 + 7 * i|^2/2 = 74/2 = 37$

5.6.11 The integer Euclidean remainder : `irem remain smod mods mod %`

`irem` (or `remain`) returns the integer remainder r from the Euclidean division of two integers a and b given as arguments ($a = b * q + r$ with $0 \leq r < b$).
For Gaussian integers, we choose q so that $b * q$ is as near to a as possible and it can be proved that r may be chosen so that $|r|^2 \leq |b|^2/2$.
Input :

```
irem(148,5)
```

Output :

```
3
```

`irem` works with long integers or with Gaussian integers.
Example :

```
irem(factorial(148),factorial(45)+2 )
```

Output :

`1116153397282299330183389178030083019921209420472396393 12`

Another example

`irem(25+12*i,5+7*i)`

Output :

`-4+i`

Here $a - b * q = -4 + i$ and $| - 4 + i|^2 = 17 < |5 + 7 * i|^2/2 = 74/2 = 37$

`smod` or `mods` is a prefixed function and has two integers a and b as arguments. `smod` or `mods` returns the symmetric remainder s of the Euclidean division of the arguments a and b ($a = b * q + s$ with $-b/2 < s \leq b/2$).

Input :

`smod(148,5)`

Output :

`-2`

`mod` (or `%`) is an infixed function and has two integers a and b as arguments. `mod` (or `%`) returns $r\%b$ of Z/bZ where r is the remainder of the Euclidean division of the arguments a and b.

Input :

`148 mod 5`

or :

`148 % 5`

Output :

`3 % 5`

Note that the answer `3 % 5` is not an integer (3) but an element of $Z/5Z$ (see 5.32 to have the possible operations in $Z/5Z$).

5.6.12 Euclidean quotient and euclidean remainder of two integers : `iquorem`

`iquorem` returns the list of the quotient q and the remainder r of the Euclidean division between two integers a and b given as arguments ($a = b * q + r$ with $0 \leq r < b$).

Input :

`iquorem(148,5)`

Output :

`[29,3]`

5.6.13 Test of evenness : `even`

`even` takes as argument an integer `n`.
`even` returns `1` if `n` is even and returns `0` if `n` is odd.
Input :

```
even(148)
```

Output :

```
1
```

Input :

```
even(149)
```

Output :

```
0
```

5.6.14 Test of oddness : `odd`

`odd` takes as argument an integer `n`.
`odd` returns `1` if `n` is odd and returns `0` if `n` is even.
Input :

```
odd(148)
```

Output :

```
0
```

Input :

```
odd(149)
```

Output :

```
1
```

5.6.15 Test of pseudo-primality : `is_pseudoprime`

If `is_pseudoprime(n)` returns `2` (true), then `n` is prime.
If it returns 1, then `n` is pseudo-prime (most probably prime).
If it returns 0, then `n` is not prime.
DEFINITION: For numbers less than 10^{14}, pseudo-prime and prime are equivalent. But for numbers greater than 10^{14}, a pseudo-prime is a number with a large probability of being prime (cf. Rabin's Algorithm and Miller-Rabin's Algorithm in the Algorithmic part (menu `Help->Manuals->Programming`)).
Input :

```
is_pseudoprime(100003)
```

Output :

```
2
```

Input :

`is_pseudoprime(9856989898997)`

Output :

`2`

Input :

`is_pseudoprime(14)`

Output :

`0`

Input :

`is_pseudoprime(9856989898997789789)`

Output :

`1`

5.6.16 Test of primality : `is_prime isprime isPrime`

`is_prime(n)` returns `1` (true) if `n` is prime and `0` (false) if `n` is not prime.
`isprime` returns `true` or `false`.
Use the command `pari("isprime",n,1)` to have a primality certificate (see the documentation PARI/GP with the menu `Help->Manuals->PARI-GP`) and `pari("isprime",n,2)` to use the APRCL test.

Input :

`is_prime(100003)`

Output :

`1`

Input :

`isprime(100003)`

Output :

`true`

Input :

`is_prime(98569898989987)`

Output :

`1`

Input :

`is_prime(14)`

Output :

```
0
```

Input :

```
isprime(14)
```

Output :

```
false
```

Input :

```
pari("isprime",9856989898997789789,1)
```

This returns the coefficients giving the proof of primality by the $p-1$ Selfridge-Pocklington-Lehmer test :

```
[[2,2,1],[19,2,1],[941,2,1],[1873,2,1],[94907,2,1]]
```

Input :

```
isprime(9856989898997789789)
```

Output :

```
true
```

5.6.17 The smallest pseudo-prime greater than `n` : `nextprime`

`nextprime(n)` returns the smallest pseudo-prime (or prime) greater than `n`.
Input :

```
nextprime(75)
```

Output :

```
79
```

5.6.18 The greatest pseudo-prime less than `n` : `prevprime`

`prevprime(n)` returns the greatest pseudo-prime (or prime) less than `n`.
Input :

```
prevprime(75)
```

Output :

```
73
```

5.6.19 The n-th prime number : ithprime

`ithprime(n)` returns the n-th prime number less than 10000 (current limitation).
Input :

`ithprime(75)`

Output :

`379`

Input :

`ithprime(1229)`

Output :

`9973`

Input :

`ithprime(1230)`

Output :

`ithprime(1230)`

because `ithprime(1230)` is greater than 10000.

5.6.20 Bézout's Identity : iegcd igcdex

`iegcd(a,b)` or `igcdex(a,b)` returns the coefficients of the Bézout's Identity for two integers given as arguments.
`iegcd(a,b)` or `igcdex(a,b)` returns `[u,v,d]` such that `au+bv=d` and `d=gcd(a,b)`.
Input :

`iegcd(48,30)`

Output :

`[2,-3,6]`

In other words :

$$2 \cdot 48 + (-3) \cdot 30 = 6$$

5.6.21 Solving au+bv=c in $\mathbb{Z}$: iabcuv

`iabcuv(a,b,c)` returns `[u,v]` so that `au+bv=c`.
`c` must be a multiple of `gcd(a,b)` for the existence of a solution.
Input :

`iabcuv(48,30,18)`

Output :

`[6,-9]`

5.6.22 Chinese remainders : `ichinrem, ichrem`

`ichinrem([a,p],[b,q])` or `ichrem([a,p],[b,q])` returns a list `[c,lcm(p,q)]` of 2 integers.
The first number `c` is such that

$$\forall k \in \mathbb{Z}, \quad d = c + k \times \text{lcm}(p, q)$$

has the properties

$$d = a \pmod p, \quad d = b \pmod q$$

If `p` and `q` are coprime, a solution `d` always exists and all the solutions are congruent modulo `p*q`.
Examples :
Solve :

$$\begin{cases} x &= 3 \pmod 5 \\ x &= 9 \pmod{13} \end{cases}$$

Input :

```
ichinrem([3,5],[9,13])
```

or :

```
ichrem([3,5],[9,13])
```

Output :

```
[-17,65]
```

so `x=-17 (mod 65)`
We can also input :

```
ichrem(3%5,9%13)
```

Output :

```
-17%65
```

Solve :

$$\begin{cases} x &= 3 \pmod 5 \\ x &= 4 \pmod 7 \\ x &= 1 \pmod 9 \end{cases}$$

First input :

```
tmp:=ichinrem([3,5],[4,7])
```

or :

```
tmp:=ichrem([3,5],[4,7])
```

Output :

```
[-17,35]
```

Then input :

```
ichinrem([1,9],tmp)
```

or :

```
ichrem([1,9],tmp)
```

Output :

```
[-17,315]
```

hence `x=-17 (mod 315)`
Alternative input:

```
ichinrem([3%5,4%7,1%9])
```

Output :

```
-17%315
```

Remark
`ichrem` (or `ichinrem`) may be used to find the coefficients of a polynomial whose equivalence classes are known modulo several integers, for example find $ax + b$ modulo $315 = 5 \times 7 \times 9$ under the assumptions:

$$\left\{\begin{array}{ll} a = & 3 \pmod 5 \\ a = & 4 \pmod 7 \\ a = & 1 \pmod 9 \end{array}\right. , \quad \left\{\begin{array}{ll} b = & 1 \pmod 5 \\ b = & 2 \pmod 7 \\ b = & 3 \pmod 9 \end{array}\right.$$

Input :

```
ichrem((3x+1)%5,(4x+2)%7,(x+3)%9)
```

Output :

```
(-17%315× x+156%315
```

hence `a=-17 (mod 315)` and `b=156 (mod 315)`.

5.6.23 Chinese remainders for lists of integers : `chrem`

`chrem` takes as argument 2 lists of integers of the same size.
`chrem` returns a list of 2 integers.
For example, `chrem([a,b,c],[p,q,r])` returns the list `[x,lcm(p,q,r)]` where `x=a mod p` and `x=b mod q` and `x=c mod r`.
A solution `x` always exists if `p, q, r` are mutually primes, and all the solutions are equal modulo `p*q*r`.
BE CAREFUL with the order of the parameters, indeed :
`chrem([a,b],[p,q])=ichrem([a,p],[b,q])=`
`ichinrem([a,p],[b,q])`
Examples :
Solve :

$$\left\{\begin{array}{ll} x = & 3 \pmod 5 \\ x = & 9 \pmod{13} \end{array}\right.$$

Input :

```
chrem([3,9],[5,13])
```

Output :

```
[-17,65]
```

so, `x=-17 (mod 65)`
Solve :

$$\begin{cases} x = & 3 \pmod 5 \\ x = & 4 \pmod 6 \\ x = & 1 \pmod 9 \end{cases}$$

Input :

```
chrem([3,4,1],[5,6,9])
```

Output :

```
[28,90]
```

so `x=28 (mod 90)`
Remark
`chrem` may be used to find the coefficients of a polynomial whose equivalence classes are known modulo several integers, for example find $ax + b$ modulo $315 = 5 \times 7 \times 9$ under the assumptions:

$$\begin{cases} a = & 3 \pmod 5 \\ a = & 4 \pmod 7 \\ a = & 1 \pmod 9 \end{cases}, \quad \begin{cases} b = & 1 \pmod 5 \\ b = & 2 \pmod 7 \\ b = & 3 \pmod 9 \end{cases}$$

Input :

```
chrem([3x+1,4x+2,x+3],[5,7,9])
```

Output :

```
[-17x+156,315]
```

hence, `a=-17 (mod 315)` and `b=156 (mod 315).`

5.6.24 Solving $a^2 + b^2 = p$ in $\mathbb{Z}$: `pa2b2`

`pa2b2` decompose a prime integer p congruent to 1 modulo 4, as a sum of squares : $p = a^2 + b^2$. The result is the list `[a,b]`.
Input :

```
pa2b2(17)
```

Output :

```
[4,1]
```

indeed $17 = 4^2 + 1^2$

5.6.25 The Euler indicatrix : `euler phi`

`euler` (or `phi`) returns the Euler indicatrix for a integer.
`euler(n)` (or `phi(n)`) is equal to the number of integers less than `n` and prime with `n`.
Input :

```
euler(21)
```

Output :

```
12
```

In other words E={2,4,5,7,8,10,11,13,15,16,17,19} is the set of integers less than 21 and coprime with 21. There are 12 members in this set, hence Cardinal(E)=12.

Euler has introduced this function to generalize the little Fermat theorem:
If a and n are mutually prime then $a^{euler(n)} = 1 \mod n$

5.6.26 Legendre symbol : `legendre_symbol`

If n is prime, we define the Legendre symbol of a written $\left(\frac{a}{n}\right)$ by :

$$\left(\frac{a}{n}\right) = \begin{cases} 0 & \text{if } a = 0 \mod n \\ 1 & \text{if } a \neq 0 \mod n \text{ and if } a = b^2 \mod n \\ -1 & \text{if } a \neq 0 \mod n \text{ and if } a \neq b^2 \mod n \end{cases}$$

Some properties

- If n is prime :
$$a^{\frac{n-1}{2}} = \left(\frac{a}{n}\right) \mod n$$
-
$$\begin{aligned} \left(\frac{p}{q}\right).\left(\frac{q}{p}\right) &= (-1)^{\frac{p-1}{2}}.(-1)^{\frac{q-1}{2}} \text{ if } p \text{ and } q \text{ are odd and positive} \\ \left(\frac{2}{p}\right) &= (-1)^{\frac{p^2-1}{8}} \\ \left(\frac{-1}{p}\right) &= (-1)^{\frac{p-1}{2}} \end{aligned}$$

`legendre_symbol` takes two arguments a and n and returns the Legendre symbol $\left(\frac{a}{n}\right)$.
Input :

```
legendre_symbol(26,17)
```

Output :

```
1
```

Input :

```
legendre_symbol(27,17)
```

Output :

```
-1
```

Input :

```
legendre_symbol(34,17)
```

Output :

```
0
```

5.6.27 Jacobi symbol : `jacobi_symbol`

If n is not prime, the Jacobi symbol of a, denoted as $\left(\frac{a}{n}\right)$, is defined from the Legendre symbol and from the decomposition of n into prime factors. Let

$$n = p_1^{\alpha_1}..p_k^{\alpha_k}$$

where p_j is prime and α_j is an integer for $j = 1..k$. The Jacobi symbol of a is defined by :

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1} ... \left(\frac{a}{p_k}\right)^{\alpha_k}$$

`jacobi_symbol` takes two arguments a and n, and it returns the Jacobi symbol $\left(\frac{a}{n}\right)$.

Input :

```
jacobi_symbol(25,12)
```

Output :

```
1
```

Input :

```
jacobi_symbol(35,12)
```

Output :

```
-1
```

Input :

```
jacobi_symbol(33,12)
```

Output :

```
0
```

5.7 Combinatorial analysis

5.7.1 Factorial : `factorial` `!`

`factorial` (prefix) or `!` (postfix) takes as argument an integer n.
`factorial(n)` or `n!` returns $n!$.
Input :

```
factorial(10)
```

or

```
10!
```

Output :

```
3628800
```

5.7.2 Binomial coefficients : `binomial` `comb` `nCr`

`comb` or `nCr` or `binomial` takes as argument two integers `n` and `p`.
`comb(n,p)` or `nCr(n,p)` or `binomial(n,p)` returns $\binom{n}{p} = C_n^p$.
Input :

```
comb(5,2)
```

Output :

```
10
```

Remark
`binomial` (unlike `comb, nCr`) may have a third real argument, in this case `binomial(n,p,a)` returns $\binom{n}{p} a^p(1-a)^{n-p}$.

5.7.3 Permutations : `perm` `nPr`

`perm` or `nPr` takes as arguments two integers n and p.
`perm(n,p)` or `nPr(n,p)` returns P_n^p.
Input :

```
perm(5,2)
```

Output :

```
20
```

5.7.4 Random integers : `rand`

`rand` takes as argument an integer n or no argument.

- `rand(n)` returns a random integer p such that $0 \leq p < n$.
  Input :

```
rand(10)
```

Output for example :

```
8
```

- `rand()` returns a random integer p such that $0 \leq p < 2^{31}$ (or on 64 bits architecture $0 \leq p < 2^{63}$).
  Input :

  ```
  rand()
  ```

  Output for example :

  ```
  846930886
  ```

5.8 Rationals

5.8.1 Transform a floating point number into a rational : `exact` `float2rational`

`float2rational` or `exact` takes as argument a floating point number `d` and returns a rational number `q` close to `d` such that `abs(d-q)<epsilon`. `epsilon` is defined in the `cas` configuration (`Cfg` menu) or with the `cas_setup` command.
Input :

```
float2rational(0.3670520231)
```

Output when `epsilon=1e-10`:

```
127/346
```

Input :

```
evalf(363/28)
```

Output :

```
12.9642857143
```

Input :

```
float2rational(12.9642857143)
```

Output :

```
363/28
```

If two representations are mixed, for example :

```
1/2+0.7
```

the rational is converted to a float, output :

```
1.2
```

Input :

```
1/2+float2rational(0.7)
```

Output :

```
6/5
```

5.8.2 Integer and fractional part : `propfrac propFrac`

`propfrac(A/B)` or `propFrac(A/B)` returns

$$q + \frac{r}{b} \text{ with } 0 \leq r < b$$

if $\frac{A}{B} = \frac{a}{b}$ with $\gcd(a, b) = 1$ and $a = bq + r$.
For rational fractions, cf. 5.29.8.
Input :

```
propfrac(42/15)
```

Output :

```
2+4/5
```

Input :

```
propfrac(43/12)
```

Output :

```
3+7/12
```

5.8.3 Numerator of a fraction after simplification : `numer getNum`

`numer` or `getNum` takes as argument a fraction and returns the numerator of this fraction after simplification (for rational fractions, see 5.29.2).
Input :

```
numer(42/12)
```

or :

```
getNum(42/12)
```

Output :

```
7
```

To avoid simplifications, the argument must be quoted (for rational fractions see 5.29.1).
Input :

```
numer('42/12')
```

or :

```
getNum('42/12')
```

Output :

```
42
```

5.8.4 Denominator of a fraction after simplification : `denom` `getDenom`

`denom` or `getDenom` takes as argument a fraction and returns the denominator of this fraction after simplification (for rational fractions see 5.29.4).
Input :

```
denom(42/12)
```

or :

```
getDenom(42/12)
```

Output :

```
2
```

To avoid simplifications, the argument must be quoted (for rational fractions see 5.29.3).
Input :

```
denom('42/12')
```

or :

```
getDenom('42/12')
```

Output :

```
12
```

5.8.5 Numerator and denominator of a fraction : `f2nd` `fxnd`

`f2nd` (or `fxnd`) takes as argument a fraction and returns the list of the numerator and denominator of this fraction after simplification (for rational fractions see 5.29.5).
Input :

```
f2nd(42/12)
```

Output :

```
[7,2]
```

5.8.6 Simplification of a pair of integers : `simp2`

`simp2` takes as argument two integers or a list of two integers which represent a fraction (for two polynomials see 5.29.6).
`simp2` returns the list of the numerator and the denominator of an irreducible representation of this fraction (i.e. after simplification).
Input :

```
simp2(18,15)
```

Output :

```
[6,5]
```

Input :

```
simp2([42,12])
```

Output :

```
[7,2]
```

5.8.7 Continued fraction representation of a real : `dfc`

`dfc` takes as argument a real or a rational or a floating point number `a` and an integer `n` (or a real `epsilon`).
`dfc` returns the list of the continued fraction representation of `a` of order `n` (or with precision `epsilon` i.e. the continued fraction representation which approximates `a` or `evalf(a)` with precision `epsilon`, by default `epsilon` is the value of the `epsilon` defined in the `cas` configuration with the menu `Cfg`▶`Cas Configuration`).
`convert` with the option `confrac` has a similar functionality: in that case the value of `epsilon` is the value of the `epsilon` defined in the `cas` configuration with the menu `Cfg`▶`Cas Configuration` (see 5.21.24) and the answer may be stored in an optional third argument.

Remarks

- If the last element of the result is a list, the representation is ultimately periodic, and the last element is the period. It means that the real is a root of an equation of order 2 with integer coefficients.
- if the last element of the result is not an integer, it represents a remainder r ($a = a0 + 1/.... + 1/an + 1/r$). Be aware that this remainder has lost most of its accuracy.

If `dfc(a)=[a0,a1,a2,[b0,b1]]` that means :

$$a = a0 + \frac{1}{a1 + \frac{1}{a2 + \frac{1}{b0 + \frac{1}{b1 + \frac{1}{b0 + ...}}}}}$$

If `dfc(a)=[a0,a1,a2,r]` that means :

$$a = a0 + \frac{1}{a1 + \frac{1}{a2 + \frac{1}{r}}}$$

Input :

```
dfc(sqrt(2),5)
```

Output :

```
[1,2,[2]]
```

Input :

```
dfc(evalf(sqrt(2)),1e-9)
```

or :

```
dfc(sqrt(2),1e-9)
```

Output :

```
[1,2,2,2,2,2,2,2,2,2,2,2,2]
```

Input :

```
convert(sqrt(2),confrac,'dev')
```

Output (if in the `cas` configuration `epsilon=1e-9`) :

```
[1,2,2,2,2,2,2,2,2,2,2,2,2]
```

and `[1,2,2,2,2,2,2,2,2,2,2,2,2]` is stored in `dev`.
Input :

```
dfc(9976/6961,5)
```

Output :

```
[1,2,3,4,5,43/7]
```

Input to verify:

```
1+1/(2+1/(3+1/(4+1/(5+7/43))))
```

Output :

```
9976/6961
```

Input :

```
convert(9976/6961,confrac,'l')
```

Output (if in the `cas` configuration `epsilon=1e-9`) :

```
[1,2,3,4,5,6,7]
```

and `[1,2,3,4,5,6,7]` is stored in `l`
Input :

```
dfc(pi,5)
```

Output :

```
[3,7,15,1,292,(-113*pi+355)/(33102*pi-103993)]
```

Input :

```
dfc(evalf(pi),5)
```

Output (if floats are hardware floats, e.g. for Digits=12) :

```
[3,7,15,1,292,1.57581843574]
```

Input :

```
dfc(evalf(pi),1e-9)
```

or :

```
dfc(pi,1e-9)
```

or (if in the `cas` configuration `epsilon=1e-9`) :

```
convert(pi,confrac,'ll')
```

Output :

```
[3,7,15,1,292]
```

5.8.8 Transform a continued fraction representation into a real : `dfc2f`

`dfc2f` takes as argument a list representing a continued fraction, namely

- a list of integers for a rational number
- a list whose last element is a list for an ultimately periodic representation, i.e. a quadratic number, that is a root of a second order equation with integer coefficients.
- or a list with a remainder r as last element ($a = a0 + 1/.... + 1/an + 1/r$).

`dfc2f` returns the rational number or the quadratic number with the argument as continued fraction representation.
Input :

```
dfc2f([1,2,[2]])
```

Output :

```
1/(1/(1+sqrt(2))+2)+1
```

After simplification with `normal` :

```
sqrt(2)
```

Input :

```
dfc2f([1,2,3])
```

Output :

```
10/7
```

Input :

```
normal(dfc2f([3,3,6,[3,6]]))
```

Output :

```
sqrt(11)
```

Input :

```
dfc2f([1,2,3,4,5,6,7])
```

Output :

```
9976/6961
```

Input to verify :

```
1+1/(2+1/(3+1/(4+1/(5+1/(6+1/7)))))
```

Output :

```
9976/6961
```

Input :

```
dfc2f([1,2,3,4,5,43/7])
```

Output :

```
9976/6961
```

Input to verify :

```
1+1/(2+1/(3+1/(4+1/(5+7/43))))
```

Output :

```
9976/6961
```

5.8.9 The n-th Bernoulli number : bernoulli

bernoulli takes as argument an integer n.
bernoulli returns the n-th Bernoulli number $B(n)$.
The Bernoulli numbers are defined by :

$$\frac{t}{e^t-1}=\sum_{n=0}^{+\infty}\frac{B(n)}{n!}t^n$$

Bernoulli polynomials B_k are defined by :

$$B_0=1, \quad {B_k}'(x)=kB_{k-1}(x), \quad \int_0^1 B_k(x)dx=0$$

and the relation $B(n)=B_n(0)$ holds.
Input :

```
bernoulli(6)
```

Output :

```
1/42
```

5.8.10 Access to PARI/GP commands: pari

- pari with a string as first argument (the PARI command name) execute the corresponding PARI command with the remaining arguments. For example pari("weber",1+i) executes the PARI command weber with the argument 1+i.
- pari without argument exports all PARI/GP functions
  - with the same command name if they are not already defined inside Xcas
  - with their original command name with the prefix pari_

  For example, after calling pari(), pari_weber(1+i) or weber(1+i) will execute the PARI command weber with the argument 1+i.

The documentation of PARI/GP is available with the menu Help->Manuals.

5.9 Real numbers

5.9.1 Eval a real at a given precision : `evalf` and `Digits`, `DIGITS`

- A real number is an exact number and its numeric evaluation at a given precision is a floating number represented in base 2.
  The precision of a floating number is the number of bits of its mantissa, which is at least 53 (hardware float numbers, also known as `double`). Floating numbers are displayed in base 10 with a number of digits controlled by the user either by assigning the `Digits` variable or by modifying the Cas configuration. By default `Digits` is equal to 12. The number of digits displayed controls the number of bits of the mantissa, if Digits is less than 15, 53 bits are used, if Digits is strictly greater than 15, the number of bits is a roundoff of Digits times the log of 10 in base 2.

- An expression is coerced into a floating number with the `evalf` command. `evalf` may have an optional second argument which will be used to evaluate with a given precision.

- Note that if an expression contains a floating number, evaluation will try to convert other arguments to floating point numbers in order to coerce the whole expression to a single floating number.

Input :

```
1+1/2
```

Output :

```
3/2
```

Input :

```
1.0+1/2
```

Output :

```
1.5
```

Input:

```
exp(pi*sqrt(20))
```

Output :

```
exp(pi*2*sqrt(5))
```

With `evalf`, input :

```
evalf(exp(pi*2*sqrt(5)))
```

Output :

```
1263794.75367
```

Input :

```
1.1^20
```

Output :

```
6.72749994933
```

Input :

```
sqrt(2)^21
```

Output :

```
sqrt(2)*2^10
```

Input for a result with 30 digits :

```
Digits:=30
```

Input for the numeric value of $e^{\pi\sqrt{163}}$:

```
evalf(exp(pi*sqrt(163)))
```

Output :

```
0.262537412640768743999999999985e18
```

Note that `Digits` is now set to 30. If you don't want to change the value of `Digits` you may input

```
evalf(exp(pi*sqrt(163)),30)
```

5.9.2 Usual infixed functions on reals : `+,-,*,/,^`

`+,-,*,/,^` are the usual operators to do additions, subtractions, multiplications, divisions and for raising to a power.
Input :

```
3+2
```

Output :

```
5
```

Input :

```
3-2
```

Output :

```
1
```

Input :

```
3*2
```

Output :

```
6
```

Input :

```
3/2
```

Output :

```
3/2
```

Input :

```
3.2/2.1
```

Output :

```
1.52380952381
```

Input :

```
3^2
```

Output :

```
9
```

Input :

```
3.2^2.1
```

Output :

```
11.5031015682
```

Remark

You may use the square key or the cube key if your keyboard has one, for example : 3^2 returns 9.

Remark on non integral powers

- If x is not an integer, then $a^x = \exp(x\ln(a))$, hence a^x is well-defined only for $a > 0$ if x is not rational. If x is rational and $a < 0$, the principal determination of the logarithm is used, leading to a complex number.
- Hence be aware of the difference between $\sqrt[n]{a}$ and $a^{\frac{1}{n}}$ when n is an odd integer.
  For example, to draw the graph of $y = \sqrt[3]{x^3 - x^2}$, input :

  ```
  plotfunc(ifte(x>0,(x^3-x^2)^(1/3),
   -(x^2-x^3)^(1/3)),x,xstep=0.01)
  ```

  You might also input :

  ```
  plotimplicit(y^3=x^3-x^2)
  ```

  but this is much slower and much less accurate.

5.9.3 Usual prefixed functions on reals : `rdiv`

`rdiv` is the prefixed form of the division function.
Input :

```
rdiv(3,2)
```

Output :

```
3/2
```

Input :

```
rdiv(3.2,2.1)
```

Output :

```
1.52380952381
```

5.9.4 n-th root : `root`

`root` takes two arguments : an integer n and a number a.
`root` returns the n-th root of a (i.e. $a^{1/n}$). If $a < 0$, the n-th root is a complex number of argument $2\pi/n$.
Input :

```
root(3,2)
```

Output :

```
2^(1/3)
```

Input :

```
root(3,2.0)
```

Output :

```
1.259921049892
```

Input :

```
root(3,sqrt(2))
```

Output :

```
2^(1/6)
```

5.9.5 Error function : `erf`

`erf` takes as argument a number a.
`erf` returns the floating point value of the error function at $x = a$, where the error function is defined by :

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

The normalization is chosen so that:

$$\text{erf}(+\infty) = 1, \quad \text{erf}(-\infty) = -1$$

since :

$$\int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

Input :

```
erf(1)
```

Output :

```
0.84270079295
```

Input :

```
erf(1/(sqrt(2)))*1/2+0.5
```

Output :

```
0.841344746069
```

Remark
The relation between `erf` and `normal_cdf` is :

$$\texttt{normal\_cdf}(x) = \frac{1}{2} + \frac{1}{2}\texttt{erf}(\frac{x}{\sqrt{2}})$$

Indeed, making the change of variable $t = u * \sqrt{2}$ in

$$\text{normal\_cdf}(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$$

gives :

$$\text{normal\_cdf}(x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{2}}} e^{-u^2} du = \frac{1}{2} + \frac{1}{2}\text{erf}(\frac{x}{\sqrt{2}})$$

Check :
`normal_cdf(1)=0.841344746069`

5.9.6 Complementary error function: `erfc`

`erfc` takes as argument a number a.
`erfc` returns the value of the complementary error function at $x = a$, this function is defined by :

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt = 1 - \text{erf}(x)$$

Hence $\text{erfc}(0) = 1$, since :

$$\int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

Input :

```
erfc(1)
```

Output :

```
0.15729920705
```

Input :

```
1- erfc(1/(sqrt(2)))*1/2
```

Output :

```
0.841344746069
```

Remark
The relation between `erfc` and `normal_cdf` is :

$$\texttt{normal\_cdf}(x) = 1 - \frac{1}{2}\texttt{erfc}(\frac{x}{\sqrt{2}})$$

Check :
`normal_cdf(1)=0.841344746069`

5.9.7 The Γ function : `Gamma`

`Gamma` takes as argument a number a.
`Gamma` returns the value of the Γ function in a, defined by :

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt, \text{ if } x > 0$$

If x is a positive integer, Γ is computed by applying the recurrence :

$$\Gamma(x + 1) = x * \Gamma(x), \quad \Gamma(1) = 1$$

Hence :

$$\Gamma(n + 1) = n!$$

Input :

```
Gamma(5)
```

Output :

```
24
```

Input :

```
Gamma(0.7)
```

Output :

```
1.29805533265
```

Input :

```
Gamma(-0.3)
```

Output :

```
-4.32685110883
```

Indeed : `Gamma(0.7)=-0.3*Gamma(-0.3)`
Input :

```
Gamma(-1.3)
```

Output :

```
3.32834700679
```

Indeed `Gamma(0.7)=-0.3*Gamma(-0.3)=(-0.3)*(-1.3)*Gamma(-1.3)`

5.9.8 The β function : `Beta`

`Beta` takes as argument two reals a, b.
`Beta` returns the value of the β function at $a, b \in \mathbb{R}$, defined by :

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x) * \Gamma(y)}{\Gamma(x+y)}$$

Remarkable values :

$$\beta(1,1) = 1, \quad \beta(n,1) = \frac{1}{n}, \quad \beta(n,2) = \frac{1}{n(n+1)}$$

`Beta(x,y)` is defined for x and y positive reals (to ensure the convergence of the integral) and by prolongation for x and y if they are not negative integers.
Input :

```
Beta(5,2)
```

Output :

```
1/30
```

Input :

```
Beta(x,y)
```

Output :

```
Gamma(x)*Gamma(y)/Gamma(x+y)
```

Input :

```
Beta(5.1,2.2)
```

Output :

```
0.0242053671402
```

5.9.9 Derivatives of the DiGamma function : `Psi`

`Psi` takes as arguments a real a and an integer n (by default $n = 0$).
`Psi` returns the value of the n-th derivative of the DiGamma function at $x = a$, where the DiGamma function is the first derivative of $\ln(\Gamma(x))$. This function is used to evaluated sums of rational functions having poles at integers.
Input :

```
Psi(3,1)
```

Output :

```
pi^2/6-5/4
```

If `n=0`, you may use `Psi(a)` instead of `Psi(a,0)` to compute the value of the DiGamma function at $x = a$.
Input :

```
Psi(3)
```

Output :

```
Psi(1)+3/2
```

Input :

```
evalf(Psi(3))
```

Output :

```
.922784335098
```

5.9.10 The ζ function : `Zeta`

`Zeta` takes as argument a real x.
`Zeta` returns for $x > 1$:

$$\zeta(x) = \sum_{n=1}^{+\infty} \frac{1}{n^x}$$

and for $x < 1$ its meromorphic continuation.
Input :

```
Zeta(2)
```

Output :

```
pi^2/6
```

Input :

```
Zeta(4)
```

Output :

```
pi^4/90
```

5.9.11 Airy functions : `Airy_Ai` and `Airy_Bi`

`Airy_Ai` and `Airy_Bi` take as arguments a real x.
`Airy_Ai` and `Airy_Bi` are two independent solutions of the equation

$$y'' - x * y = 0$$

They are defined by :

$$\begin{aligned}
\text{Airy_Ai}(x) &= (1/\pi)\int_0^{\infty} \cos(t^3/3 + x * t)dt \\
\text{Airy_Bi}(x) &= (1/\pi)\int_0^{\infty} (e^{-t^3/3} + \sin(t^3/3 + x * t))dt
\end{aligned}$$

Properties :

$$\begin{aligned}
\text{Airy_Ai(x)} &= \text{Airy_Ai}(0) * f(x) + \text{Airy_Ai}'(0) * g(x) \\
\text{Airy_Bi(x)} &= \sqrt{3}(\text{Airy_Ai}(0) * f(x) - \text{Airy_Ai}'(0) * g(x))
\end{aligned}$$

where f and g are two entire series solutions of

$$w'' - x * w = 0$$

more precisely :

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} 3^k \left(\frac{\Gamma(k+\frac{1}{3})}{\Gamma(\frac{1}{3})}\right) \frac{x^{3k}}{(3k)!} \\
g(x) &= \sum_{k=0}^{\infty} 3^k \left(\frac{\Gamma(k+\frac{2}{3})}{\Gamma(\frac{2}{3})}\right) \frac{x^{3k+1}}{(3k+1)!}
\end{aligned}$$

Input :

```
Airy_Ai(1)
```

Output :

```
0.135292416313
```

Input :

```
Airy_Bi(1)
```

Output :

```
1.20742359495
```

Input :

```
Airy_Ai(0)
```

Output :

```
0.355028053888
```

Input :

```
Airy_Bi(0)
```

Output :

```
0.614926627446
```

5.10 Permutations

A permutation p of size n is a bijection from $[0..n-1]$ on $[0..n-1]$ and is represented by the list : $[p(0), p(1), p(2)...p(n-1)]$.
For example, the permutation p represented by $[1, 3, 2, 0]$ is the application from $[0, 1, 2, 3]$ on $[0, 1, 2, 3]$ defined by :

$$p(0) = 1,\ p(1) = 3,\ p(2) = 2,\ p(3) = 0$$

A cycle c of size p is represented by the list $[a_0, ..., a_{p-1}]$ $(0 \leq a_k \leq n-1)$ it is the permutation such that

$$c(a_i) = a_{i+1} \text{ for } (i = 0..p-2), \quad c(a_{p-1}) = a_0, \quad c(k) = k \text{ otherwise}$$

A cycle c is represented by a list and a cycle decomposition is represented by a list of lists.
For example, the cycle c represented by the list $[3, 2, 1]$ is the permutation c defined by $c(3) = 2,\ c(2) = 1,\ c(1) = 3,\ c(0) = 0$ (i.e. the permutation represented by the list $[0, 3, 1, 2]$).

5.10.1 Random permutation : `randperm`

`randperm` takes as argument an integer n.
`randperm` returns a random permutation of $[0..n-1]$.
Input :

```
randperm(3)
```

Output :

```
[2,0,1]
```

5.10.2 Decomposition as a product of disjoint cycles : `permu2cycles`

`permu2cycles` takes as argument a permutation.
`permu2cycles` returns its decomposition as a product of disjoint cycles.
Input :

```
permu2cycles([1,3,4,5,2,0])
```

Output :

```
[[0,1,3,5],[2,4]]
```

In the answer the cycles of size 1 are omitted, except if $n-1$ is a fixed point of the permutation (this is required to find the value of n from the cycle decomposition).
Input :

```
permu2cycles([0,1,2,4,3,5])
```

Output :

```
[[5],[3,4]]
```

Input :

`permu2cycles([0,1,2,3,5,4])`

Output :

`[[4,5]]`

5.10.3 Product of disjoint cycles to permutation: `cycles2permu`

`cycles2permu` takes as argument a list of cycles.
`cycles2permu` returns the permutation (of size n chosen as small as possible) that is the product of the given cycles (it is the inverse of `permu2cycles`).
Input :

`cycles2permu([[1,3,5],[2,4]])`

Output :

`[0,3,4,5,2,1]`

Input :

`cycles2permu([[2,4]])`

Output :

`[0,1,4,3,2]`

Input :

`cycles2permu([[5],[2,4]])`

Output :

`[0,1,4,3,2,5]`

5.10.4 Transform a cycle into permutation : `cycle2perm`

`cycle2perm` takes on cycle as argument.
`cycle2perm` returns the permutation of size n corresponding to the cycle given as argument, where n is chosen as small as possible (see also `permu2cycles` and `cycles2permu`).
Input :

`cycle2perm([1,3,5])`

Output :

`[0,3,2,5,4,1]`

5.10.5 Transform a permutation into a matrix : `permu2mat`

`permu2mat` takes as argument a permutation p of size n.
`permu2mat` returns the matrix of the permutation, that is the matrix obtained by permuting the rows of the identity matrix of size n with the permutation p.
Input :

```
permu2mat([2,0,1])
```

Output :

```
[[0,0,1],[1,0,0],[0,1,0]]
```

5.10.6 Checking for a permutation : `is_permu`

`is_permu` is a boolean function.
`is_permu` takes as argument a list.
`is_permu` returns 1 if the argument is a permutation and returns 0 if the argument is not a permutation.
Input :

```
is_permu([2,1,3])
```

Output :

```
0
```

Input :

```
is_permu([2,1,3,0])
```

Output :

```
1
```

5.10.7 Checking for a cycle : `is_cycle`

`is_cycle` is a boolean function.
`is_cycle` takes a list as argument.
`is_cycle` returns 1 if the argument is a cycle and returns 0 if the argument is not a cycle.
Input :

```
is_cycle([2,1,3])
```

Output :

```
1
```

Input :

```
is_cycle([2,1,3,2])
```

Output :

```
0
```

5.10.8 Product of two permutations : `p1op2`

`p1op2` takes as arguments two permutations.
`p1op2` returns the permutation obtained by composition :

$$1^{\text{st}}\text{arg} \circ 2^{\text{nd}}\text{arg}$$

Input :

```
p1op2([3,4,5,2,0,1],[2,0,1,4,3,5])
```

Output :

```
[5,3,4,0,2,1]
```

Warning
Composition is done using the standard mathematical notation, that is the permutation given as the second argument is performed first.

5.10.9 Composition of a cycle and a permutation : `c1op2`

`c1op2` takes as arguments a cycle and a permutation.
`c1op2` returns the permutation obtained by composition :

$$1^{\text{st}}\text{arg} \circ 2^{\text{nd}}\text{arg}$$

Input :

```
c1op2([3,4,5],[2,0,1,4,3,5])
```

Output :

```
[2,0,1,5,4,3]
```

Warning
Composition is done using the standard mathematical notation, that is the permutation given as the second argument is performed first.

5.10.10 Composition of a permutation and a cycle : `p1oc2`

`p1oc2` takes as arguments a permutation and a cycle.
`p1oc2` returns the permutation obtained by composition :

$$1^{\text{st}}\text{arg} \circ 2^{\text{nd}}\text{arg}$$

Input :

```
p1oc2([3,4,5,2,0,1],[2,0,1])
```

Output :

```
[4,5,3,2,0,1]
```

Warning
Composition is done using the standard mathematical notation, that is the cycle given as second argument is performed first.

5.10.11 Product of two cycles : `c1oc2`

`c1oc2` takes as arguments two cycles.
`c1oc2` returns the permutation obtained by composition :

$$1^{\text{st}}\text{arg} \circ 2^{\text{nd}}\text{arg}$$

Input :

```
c1oc2([3,4,5],[2,0,1])
```

Output :

```
[1,2,0,4,5,3]
```

Warning
Composition is done using the standard mathematical notation, that is the cycle given as second argument is performed first.

5.10.12 Signature of a permutation : `signature`

`signature` takes as argument a permutation.
`signature` returns the signature of the permutation given as argument.
The signature of a permutation is equal to :

- 1 if the permutation is equal to an even product of transpositions,
- -1 if the permutation is equal to an odd product of transpositions.

The signature of a cycle of size k is : $(-1)^{k+1}$.
Input :

```
signature([3,4,5,2,0,1])
```

Output :

```
-1
```

Indeed `permu2cycles([3,4,5,2,0,1])=[[0,3,2,5,1,4]]`.

5.10.13 Inverse of a permutation : `perminv`

`perminv` takes as argument a permutation.
`perminv` returns the permutation that is the inverse of the permutation given as argument.
Input :

```
perminv([1,2,0])
```

Output

```
[2,0,1]
```

5.10.14 Inverse of a cycle : `cycleinv`

`cycleinv` takes as argument a cycle.
`cycleinv` returns the cycle that is the inverse of the cycle given as argument.
Input :

```
cycleinv([2,0,1])
```

Output

```
[1,0,2]
```

5.10.15 Order of a permutation : `permuorder`

`permuorder` takes as argument a permutation.
`permuorder` returns the order k of the permutation p given as argument, that is the smallest integer m such that p^m is the identity.
Input :

```
permuorder([0,2,1])
```

Output

```
2
```

Input :

```
permuorder([3,2,1,4,0])
```

Output

```
6
```

5.10.16 Group generated by two permutations : `groupermu`

`groupermu` takes as argument two permutations `a` and `b`.
`groupermu` returns the group of the permutations generated by `a` and `b`.
Input :

```
groupermu([0,2,1,3],[3,1,2,0])
```

Output

```
[[0,2,1,3],[3,1,2,0],[0,1,2,3],[3,2,1,0]]
```

5.11 Complex numbers

Note that complex numbers are also used to represent a point in the plane or a 1-d function graph.

5.11.1 Usual complex functions : `+,-,*,/,^`

`+,-,*,/,^` are the usual operators to perform additions, subtractions, multiplications, divisions and for raising to an integer or a fractional power.
Input :

```
(1+2*i)^2
```

Output :

```
-3+4*i
```

5.11.2 Real part of a complex number : `re real`

`re` (or `real`) takes as argument a complex number (resp. a point A).
`re` (or `real`) returns the real part of this complex number (resp. the projection on the x axis of A).
Input :

```
re(3+4*i)
```

Output :

```
3
```

5.11.3 Imaginary part of a complex number : `im imag`

`im` (or `imag`) takes as argument a complex number (resp. a point A).
`im` (or `imag`) returns imaginary part of this complex number (resp. the projection on the y axis of A).
Input :

```
im(3+4*i)
```

Output :

```
4
```

5.11.4 Write a complex as `re(z)+i*im(z)` : `evalc`

`evalc` takes as argument a complex number `z`.
`evalc` returns this complex number, written as `re(z)+i*im(z)`.
Input :

```
evalc(sqrt(2)*exp(i*pi/4))
```

Output :

```
1+i
```

5.11.5 Modulus of a complex number : abs

abs takes as argument a complex number.
abs returns the modulus of this complex number.
Input :

```
abs(3+4*i)
```

Output :

```
5
```

5.11.6 Argument of a complex number : arg

arg takes as argument a complex number.
arg returns the argument of this complex number.
Input :

```
arg(3+4*i)
```

Output :

```
atan(4/3)
```

5.11.7 The normalized complex number : normalize unitV

normalize or unitV takes as argument a complex number.
normalize or unitV returns the complex number divided by the modulus of this complex number.
Input :

```
normalize(3+4*i)
```

Output :

```
(3+4*i)/5
```

5.11.8 Conjugate of a complex number : conj

conj takes as argument a complex number.
conj returns the complex conjugate of this complex number.
Input :

```
conj(3+4*i)
```

Output :

```
3-4*i
```

5.11.9 Multiplication by the complex conjugate : mult_c_conjugate

mult_c_conjugate takes as argument an complex expression.
If this expression has a complex denominator, mult_c_conjugate multiplies the numerator and the denominator of this expression by the complex conjugate of the denominator.
If this expression does not have a complex denominator, mult_c_conjugate multiplies the numerator and the denominator of this expression by the complex conjugate of the numerator.
Input :

```
mult_c_conjugate((2+i)/(2+3*i))
```

Output :

```
(2+i)*(2+3*(-i))/((2+3*(i))*(2+3*(-i)))
```

Input :

```
mult_c_conjugate((2+i)/2)
```

Output :

```
(2+i)*(2+-i)/(2*(2+-i))
```

5.11.10 Barycenter of complex numbers : barycentre

See also : ?? and **??**.
barycentre takes as argument two lists of the same size (resp. a matrix with two columns):

- the elements of the first list (resp. column) are points A_j or complex numbers a_j (the affixes of the points),
- the elements of the second list (resp. column) are real coefficients α_j such that $\sum \alpha_j \neq 0$.

barycentre returns the barycenter point of the points A_j weighted by the real coefficients α_j. If $\sum \alpha_j = 0$, barycentre returns an error.
Warning To have a complex number in the output, the input must be :
affix(barycentre(...,...)) because barycentre(...,...) returns a point, not a complex number.
Input :

```
affix(barycentre([1+i,1-i],[1,1]))
```

or :

```
affix(barycentre([[1+i,1],[1-i,1]]))
```

Output :

```
i
```

5.12 Algebraic expressions

5.12.1 Evaluate an expression : `eval`

`eval` is used to evaluate an expression. Since `Xcas` always evaluate expressions entered in the command line, `eval` is mainly used to evaluate a sub-expression in the equation writer.
Input :

```
a:=2
```

Output :

```
2
```

Input :

```
eval(2+3*a)
```

or

```
2+3*a
```

Output :

```
8
```

5.12.2 Evaluate algebraic expressions : `evala`

In Maple, `evala` is used to evaluate an expression with algebraic extensions. In `Xcas`, `evala` is not necessary, it behaves like `eval`.

5.12.3 Prevent evaluation : `quote hold '`

A quoted subexpression (either with `'` or with the `quote` or `hold`) command will not be evaluated.
Remark `a:=quote(a)` (or `a:=hold(a)`) is equivalent to `purge(a)` (for the sake of Maple compatibility). It returns the value of this variable (or the hypothesis done on this variable).
Input :

```
a:=2;quote(2+3*a)
```

or

```
a:=2;'2+3*a'
```

Output :

```
(2,2+3*a)
```

5.12.4 Force evaluation : `unquote`

`unquote` is used to evaluate inside a quoted expression.
For example in an affectation, the variable is automatically quoted (not evaluated) so that the user does not have to quote it explicitly each time he want to modify its value. In some circumstances, you might however want to evaluate it.
Input:

```
purge(b);a:=b;unquote(a):=3
```

Output :

```
b contains 3, hence a evals to 3
```

5.12.5 Distribution : `expand fdistrib`

`expand` or `fdistrib` takes as argument an expression.
`expand` or `fdistrib` returns the expression where multiplication is distributed with respect to the addition.
Input :

```
expand((x+1)*(x-2))
```

or :

```
fdistrib((x+1)*(x-2))
```

Output :

```
x^2-2*x+x-2
```

5.12.6 Canonical form : `canonical_form`

`canonical_form` takes as argument a trinomial of second degree.
`canonical_form` returns the canonical form of the argument.
Example :
Find the canonical form of :

$$x^2 - 6x + 1$$

Input :

```
canonical_form(x^2-6*x+1)
```

Output :

```
(x-3)^2-8
```

5.12.7 Multiplication by the conjugate quantity : `mult_conjugate`

`mult_conjugate` takes as argument an expression with a denominator or a numerator supposed to contain a square root :

- if the denominator contains a square root,
  `mult_conjugate` multiplies the numerator and the denominator of the expression by the conjugate quantity of the denominator.
- otherwise, if the numerator contains a square root,
  `mult_conjugate` multiplies the numerator and the denominator of this expression by the conjugate quantity of the numerator.

Input :

```
mult_conjugate((2+sqrt(2))/(2+sqrt(3)))
```

Output :

```
(2+sqrt(2))*(2-sqrt(3))/((2+sqrt(3))*(2-sqrt(3)))
```

Input :

```
mult_conjugate((2+sqrt(2))/(sqrt(2)+sqrt(3)))
```

Output :

```
(2+sqrt(2))*(-sqrt(2)+sqrt(3))/
((sqrt(2)+sqrt(3))*(-sqrt(2)+sqrt(3)))
```

Input :

```
mult_conjugate((2+sqrt(2))/2)
```

Output :

```
(2+sqrt(2))*(2-sqrt(2))/(2*(2-sqrt(2)))
```

5.12.8 Separation of variables : `split`

`split` takes two arguments : an expression depending on two variables and the list of these two variables.
If the expression may be factorized into two factors where each factor depends only on one variable, `split` returns the list of this two factors, otherwise it returns the list `[0]`.
Input :

```
split((x+1)*(y-2),[x,y])
```

or :

```
split(x*y-2*x+y-2,[x,y])
```

Output :

```
[x+1,y-2]
```

Input :

```
split((x^2*y^2-1,[x,y])
```

Output :

```
[0]
```

5.12.9 Factorization : `factor`

`factor` takes as argument an expression.
`factor` factorizes this expression on the field of its coefficients, with the addition of i in complex mode. If `sqrt` is enabled in the Cas configuration, polynomials of order 2 are factorized in complex mode or in real mode if the discriminant is positive.
Examples

1. Factorize $x^4 - 1$ over $\mathbb{Q}$.
   Input :

   ```
   factor(x^4-1)
   ```

   Output :

   ```
   (x^2+1)*(x+1)*(x-1)
   ```

   The coefficients are rationals, hence the factors are polynomials with rationals coefficients.

2. Factorize $x^4 - 1$ over $\mathbb{Q}[i]$
   To have a complex factorization, check `complex` in the `cas` configuration (red button displaying the status line).
   Input :

   ```
   factor(x^4-1)
   ```

   Output :

   ```
   -i*(-x+-i)*(i*x+1)*(-x+1)*(x+1)
   ```

3. Factorize $x^4 + 1$ over $\mathbb{Q}$
   Input :

   ```
   factor(x^4+1)
   ```

   Output :

   ```
   x^4+1
   ```

Indeed $x^4 + 1$ has no factor with rational coefficients.

4. Factorize $x^4 + 1$ over $\mathbb{Q}[i]$
   Check `complex` in the `cas` configuration (red button rouge displaying the status line).
   Input :

   ```
   factor(x^4-1)
   ```

   Output :

   ```
   (x^2+i)*(x^2+-i)
   ```

5. Factorize $x^4 + 1$ over $\mathbb{R}$.
   You have to provide the square root required for extending the rationals. In order to do that with the help of `Xcas`, first check `complex` in the `cas` configuration and input :

   ```
   solve(x^4+1,x)
   ```

   Output :

   ```
   [sqrt(2)/2+(i)*sqrt(2)/2,sqrt(2)/2+(i)*(-(sqrt(2)/2)),
   -sqrt(2)/2+(i)*sqrt(2)/2,-sqrt(2)/2+(i)*(-(sqrt(2)/2))]
   ```

   The roots depends on $\sqrt{2}$. Uncheck complex mode in the Cas configuration and input :

   ```
   factor(x^4+1,sqrt(2))
   ```

   Output :

   ```
   (x^2+sqrt(2)*x+1)*(x^2+(-(sqrt(2)))*x+1)
   ```

   To factorize over $\mathbb{C}$, check `complex` in the `cas` configuration or input `cFactor(x^4+1,sqrt(2))` (cf `cFactor`).

5.12.10 Complex factorization : `cFactor`

`cFactor` takes as argument an expression.
`cFactor` factorizes this expression on the field $\mathbb{Q}[i] \subset \mathbb{C}$ (or over the complexified field of the coefficients of the argument) even if you are in real mode.
Examples

1. Factorize $x^4 - 1$ over $\mathbb{Z}[i]$.
   Input :

```
cFactor(x^4-1)
```

Output :

```
-((x+-i)*((-i)*x+1)*((-i)*x+i)*(x+1))
```

2. Factorize $x^4 + 1$ over $\mathbb{Z}[i]$.
   Input :

   ```
   cFactor(x^4+1)
   ```

   Output :

   ```
   (x^2+i)*(x^2+-i)
   ```

3. For a complete factorization of $x^4 + 1$, check the sqrt box in the Cas configuration or input :

   ```
   cFactor(x^4+1,sqrt(2))
   ```

   Output :

   ```
   sqrt(2)*1/2*(sqrt(2)*x+1-i)*(sqrt(2)*x-1+i)*sqrt(2)*
           1/2*(sqrt(2)*x+1+i)*(sqrt(2)*x-1-i)
   ```

5.12.11 Zeros of an expression : `zeros`

`zeros` takes as argument an expression depending on x.
`zeros` returns a list of values of x where the expression vanishes. The list may be incomplete in exact mode if the expression is not polynomial or if intermediate factorizations have irreducible factors of order strictly greater than 2.
In real mode, (complex box unchecked in the Cas configuration or `complex_mode:=0`), only reals zeros are returned. In (`complex_mode:=1`) reals and complex zeros are returned. See also `cZeros` to get complex zeros in real mode.
Input in real mode :

```
zeros(x^2+4)
```

Output :

```
[]
```

Input in complex mode :

```
zeros(x^2+4)
```

Output :

```
[-2*i,2*i]
```

Input in real mode :

```
zeros(ln(x)^2-2)
```

Output :

```
[exp(sqrt(2)),exp(-(sqrt(2)))]
```

Input in real mode :

```
zeros(ln(y)^2-2,y)
```

Output :

```
[exp(sqrt(2)),exp(-(sqrt(2)))]
```

Input in real mode :

```
zeros(x*(exp(x))^2-2*x-2*(exp(x))^2+4)
```

Output :

```
[[log(sqrt(2)),2]
```

5.12.12 Complex zeros of an expression : cZeros

cZeros takes as argument an expression depending on x.
cZeros returns a list of complex values of x where the expression vanishes. The list may be incomplete in exact mode if the expression is not polynomial or if intermediate factorizations have irreducible factors of order strictly greater than 2.
Input in real or complex mode :

```
cZeros(x^2+4)
```

Output :

```
[-2*i,2*i]
```

Input :

```
cZeros(ln(x)^2-2)
```

Output :

```
[exp(sqrt(2)),exp(-(sqrt(2)))]
```

Input :

```
cZeros(ln(y)^2-2,y)
```

Output :

```
[exp(sqrt(2)),exp(-(sqrt(2)))]
```

Input :

```
cZeros(x*(exp(x))^2-2*x-2*(exp(x))^2+4)
```

Output :

```
[log(sqrt(2)),log(-sqrt(2)),2]
```

5.12.13 Normal form : `normal`

`normal` takes as argument an expression. The expression is considered as a rational fraction with respect to generalized identifiers (either true identifiers or transcendental functions replaced by a temporary identifiers) with coefficients in $\mathbb{Q}$ or $\mathbb{Q}[i]$ or in an algebraic extension (e.g. $\mathbb{Q}[\sqrt{2}]$). `normal` returns the expanded irreducible representation of this rational fraction. See also `ratnormal` for pure rational fractions or `simplify` if the transcendental functions are not algebraically independent.
Input :

```
normal((x-1)*(x+1))
```

Output :

```
x^2-1
```

Remarks

- Unlike `simplify`, `normal` does not try to find algebraic relations between transcendental functions like $\cos(x)^2 + \sin(x)^2 = 1$.
- It is sometimes necessary to run the `normal` command twice to get a fully irreducible representation of an expression containing algebraic extensions.

5.12.14 Simplify : `simplify`

`simplify` simplifies an expression. It behaves like `normal` for rational fractions and algebraic extensions. For expressions containing transcendental functions, `simplify` tries first to rewrite them in terms of algebraically independent transcendental functions. For trigonometric expressions, this requires radian mode (check `radian` in the `cas` configuration or input `angle_radian:=1`).
Input :

```
simplify((x-1)*(x+1))
```

Output :

```
x^2-1
```

Input :

```
simplify(3-54*sqrt(1/162))
```

Output :

```
-3*sqrt(2)+3
```

Input :

```
simplify((sin(3*x)+sin(7*x))/sin(5*x))
```

Output :

```
4*(cos(x))^2-2
```

5.12.15 Normal form for rational fractions : `ratnormal`

`ratnormal` rewrites an expression using its irreducible representation. The expression is viewed as a multivariate rational fraction with coefficients in $\mathbb{Q}$ (or $\mathbb{Q}[i]$). The variables are generalized identifiers which are assumed to be algebraically independent. Unlike with `normal`, an algebraic extension is considered as a generalized identifier. Therefore `ratnormal` is faster but might miss some simplifications if the expression contains radicals or algebraically dependent transcendental functions.
Input :

```
ratnormal((x^3-1)/(x^2-1))
```

Output :

```
(x^2+x+1)/(x+1)
```

Input :

```
ratnormal((-2x^3+3x^2+5x-6)/(x^2-2x+1))
```

Output :

```
(-2*x^2+x+6)/(x-1)
```

5.12.16 Substitute a variable by a value : `subst`

`subst` takes two or three arguments :

- an expression depending on a variable, an equality (variable=value of substitution) or a list of equalities.
- an expression depending on a variable, a variable or a list of variables, a value or a list of values for substitution.

`subst` returns the expression with the substitution done. Note that `subst` does not quote its argument, hence in a normal evaluation process, the substitution variable should be purged otherwise it will be replaced by its assigned value before substitution is done.
Input :

```
subst(a^2+1,a=2)
```

or :

```
subst(a^2+1,a,2)
```

Output (if the variable `a` is purged else first input `purge(a)`) :

```
5
```

Input :

```
subst(a^2+b,[a,b],[2,1])
```

or :

```
subst(a^2+b,[a=2,b=1])
```

Output (if the variables a and b are purged else first input purge(a,b)) :

```
5
```

subst may also be used to make a change of variable in an integral. In this case the integrate command should be quoted (otherwise, the integral would be computed before substitution) or the inert form Int should be used. In both cases, the name of the integration variable must be given as argument of Int or integrate even you are integrating with respect to x.
Input :

```
subst('integrate(sin(x^2)*x,x,0,pi/2)',x=sqrt(t))
```

or :

```
subst(Int(sin(x^2)*x,x,0,pi/2),x=sqrt(t))
```

Output

```
integrate(sin(t)*sqrt(t)*1/2*1/t*sqrt(t),t,0,(pi/2)^2)
```

Input :

```
subst('integrate(sin(x^2)*x,x)',x=sqrt(t))
```

or :

```
subst(Int(sin(x^2)*x,x),x=sqrt(t))
```

Output

```
integrate(sin(t)*sqrt(t)*1/2*1/t*sqrt(t),t)
```

5.12.17 Substitute a variable by a value (Maple and Mupad compatibility) : subs

In Maple and in Mupad, one would use the subs command to substitute a variable by a value in an expression. But the order of the arguments differ between Maple and Mupad. Therefore, to achieve compatibility, Xcas subs command arguments order depends on the mode

- In Maple mode, subs takes two arguments : an equality (variable=substitution value) and the expression.
  To substitute several variables in an expression, use a list of equality (variable names = substitution value) as first argument.
- In Mupad or Xcas or TI, subs takes two or three arguments : an expression and an equality (variable=substitution value) or an expression, a variable name and the substitution value.
  To substitute several variables, subs takes two or three arguments :
  - an expression of variables and a list of (variable names = substitution value),

  - an expression of variables, a list of variables and a list of their substitution values.

`subs` returns the expression with the substitution done. Note that `subs` does not quote its argument, hence in a normal evaluation process, the substitution variable should be purged otherwise it will be replaced by its assigned value before substitution is done.
Input in `Maple` mode (if the variable `a` is purged else input `purge(a)`) :

`subs(a=2,a^2+1)`

Output

`2^2+1`

Input in `Maple` mode (if the variables `a` and `b` are purged else input `purge(a,b)`):

`subs([a=2,b=1],a^2+b)`

Output :

`2^2+1`

Input :

`subs(a^2+1,a=2)`

or :

`subs(a^2+1,a,2)`

Output (if the variable `a` is purged else input `purge(a)`) :

`5`

Input :

`subs(a^2+b,[a=2,b=1])`

or :

`subs(a^2+b,[a,b],[2,1])`

Output (if the variables `a` and `b` are purged else input `purge(a,b)`) :

`2^2+1`

5.12.18 Evaluate a primitive at boundaries: `preval`

`preval` takes three arguments : an expression `F` depending on the variable `x`, and two expressions `a` and `b`.
`preval` computes $F_{|x=b} - F_{|x=a}$.
`preval` is used to compute a definite integral when the primitive F of the integrand f is known. Assume for example that `F:=int(f,x)`, then `preval(F,a,b)` is equivalent to `int(f,x,a,b)` but does not require to compute again `F` from `f` if you change the values of a or b.
Input :

`preval(x^2+x,2,3)`

Output :

`6`

5.12.19 Sub-expression of an expression : `part`

`part` takes two arguments : an expression and an integer n.
`part` evaluate the expression and then returns the n-th sub-expression of this expression.
Input :

```
part(x^2+x+1,2)
```

Output :

```
x
```

Input :

```
part(x^2+(x+1)*(y-2)+2,2)
```

Output :

```
(x+1)*(y-2)
```

Input :

```
part((x+1)*(y-2)/2,2)
```

Output :

```
y-2
```

5.13 Values of u_n

5.13.1 Array of values of a sequence : `tablefunc`

`tablefunc` is a command that should be used inside a spreadsheet (opened with `Alt+t`), it returns a template to fill two columns, with the table of values of a function. If the step value is 1, `tablefunc(ex,n,n0,1)`, where `ex` is an expression depending on `n`, will fill the spreadsheet with the values of the sequence $u_n = ex$ for $n = n0,\ n0 + 1,\ n0 + 2, ...$

Example : display the values of the sequence $u_n = \sin(n)$
Select a cell of a spreadsheet (for example `C0`) and input in the command line :

```
tablefunc(sin(n),n,0,1)
```

Output :

```
two columns :  n and sin(n)
```

- in the column C: the variable name `n`, the value of the step (this value should be equal to 1 for a sequence), the value of `n0` (here 0), then a recurrence formula (`C2+C$1`, ...).
- in the column D: `sin(n)`, `"Tablefunc"`, then a recurrence formula.
- For each row, the values of the sequence $\mathtt{u_n} = \sin(\mathtt{n})$ correspond to the values of `n` starting from `n=n0` (here 0).

5.13.2 Table of values and graph of a recurrent sequence : `tableseq` and `plotseq`

`tableseq` is a command that should be used inside a spreadsheet (opened with `Alt+t`), it returns a template to fill one column with $u_0,\ u_{n+1} = f(u_n)$ (one-term recurrence) or more generally $u_0, ..., u_k,\ \ u_{n+k+1} = f(u_n, u_{n+1}, ..., u_{n+k})$. The template fills the column starting from the selected cell, or starting from 0 if the whole column was selected.
See also `plotseq` (section 6.13) for a graphic representation of a one-term recurrence sequence.

Examples :

- display the values of the sequence $u_0 = 3.5,\ u_n = \sin(u_{n-1})$
  Select a cell of the spreadsheet (for example `B0`) and input in the command line :

  ```
  tableseq(sin(n),n,3.5)
  ```

  Output :

  ```
  a column with sin(n), n, 3.5 and the formula
             evalf(subst(B$0,B$1,B2))
  ```

  You get the values of the sequence $u_0 = 3.5,\ u_n = \sin(u_{n-1})$ in the column `B`.

- display the values of the Fibonacci sequence $u_0 = 1, u_1 = 1\ u_{n+2} = u_n + u_{n+1}$
  Select a cell, say `B0`, and input in the command line

  ```
  tableseq(x+y,[x,y],[1,1])
  ```

  This fills the B column sheet with

row	B
0	x+y
1	x
2	y
3	1
4	1
5	2
..	..
7	5
..	..

5.14 Operators or infixed functions

An operator is an infixed function.

5.14.1 Usual operators :`+, -, *, /, ^`

`+, -, *, /, ^` are the operators to do additions, subtractions, multiplications, divisions and for raising to a power.

5.14.2 `Xcas` operators

- `$` is the infixed version of `seq`, for example :
  `(2^k)$(k=0..3)= seq(2^k,k=0..3)=(1,2,4,8)` (do not forget to put parenthesis around the arguments),
- `mod` or `%` to define a modular number,
- `@` to compose functions for example : `(f@g)(x)=f(g(x))`,
- `@@` to compose a function many times (like a power, replacing multiplication by composition), for example : `(f@@3)(x)=f(f(f(x)))`,
- `minus union intersect` to get the difference, the union and the intersection of two sets,
- `->` to define a function,
- `:= =>` to store an expression in a variable (it is the infixed version of `sto` and the argument order is permuted for `:=`), for example : `a:=2` or `2=>a` or `sto(2,a)`.
- `=<` to store an expression in a variable, but the storage is done by reference if the target is a matrix element or a list element. This is faster if you modify objects inside an existing list or matrix of large size, because no copy is made, the change is done in place. Use with care, all objects pointing to this matrix or list will be modified.

5.14.3 Define an operator: `user_operator`

`user_operator` takes as argument :

- a string : the name of the operator,
- a function of two variables with values in $\mathbb{R}$ or in `true, false`,
- an option `Binary` for the definition or `Delete` to delete this definition.

`user_operator` returns 1 if the definition is done and else returns 0.

Example 1
Let R be defined on $\mathbb{R}$ by $x\ R\ y = x*y+x+y$.
To define the law R, input :

```
user_operator("R",(x,y)->x*y+x+y,Binary)
```

Output :

1

Input :

```
5 R 7
```

Do not forget to put spaces around R.
Output :

```
47
```

Example 2
Let S be defined on $\mathbb{N}$ by :
for x and y integers, $x\ S\ y$ <=> x and y are not coprime.
To define the law S, input :

```
user_operator("S",(x,y)->(gcd(x,y))!=1,Binary)
```

Output :

```
1
```

Input :

```
5 S 7
```

Do not forget to put spaces around S.
Output :

```
0
```

Input :

```
8 S 12
```

Do not forget to put spaces around S.
Output :

```
1
```

5.15 Functions and expressions with symbolic variables

5.15.1 The difference between a function and an expression

A function `f` is defined for example by :
`f(x):=x^2-1` or by `f:=x->x^2-1`
that is to say, for all x, $f(x)$ is equal to the expression $x^2 - 1$. In that case, to have the value of f for $x = 2$, input :`f(2)`.
But if the input is `g:=x^2-1`, then `g` is a variable where the expression $x^2 - 1$ is stored. In that case, to have the value of g for $x = 2$, input : `subst(g,x=2)` (g is an expression depending on x).

When a command expects a function as argument, this argument should be either the definition of the function (e.g. `x->x^2-1`) or a variable name assigned to a function (e.g. `f` previously defined by e.g. `f(x):=x^2-1`).
When a command expects an expression as argument, this argument should be either the definition of the expression (for example `x^2-1`), or a variable name assigned to an expression (e.g. `g` previously defined, for example, by `g:=x^2-1`), or the evaluation of a function. e.g. `f(x)` if `f` is a previously defined function, for example, by `f(x):=x^2-1`).

5.15.2 Transform an expression into a function : `unapply`

`unapply` is used to transform an expression into a function.
`unapply` takes two arguments an expression and the name of a variable.
`unapply` returns the function defined by this expression and this variable.

Warning when a function is defined, the right member of the assignment is not evaluated, hence `g:=sin(x+1); f(x):=g` does not defined the function $f : x \to \sin(x+1)$ but defines the function $f : x \to g$. To defined the former function, `unapply` should be used, like in the following example:
Input :

```
g:= sin(x+1); f:=unapply(g,x)
```

Output :

```
(sin(x+1), (x)->sin(x+1))
```

hence, the variable `g` is assigned to a symbolic expression and the variable `f` is assigned to a function.
Input :

```
unapply(exp(x+2),x)
```

Output :

```
(x)->exp(x+2)
```

Input :

```
f:=unapply(lagrange([1,2,3],[4,8,12]),x)
```

Output :

```
(x)->4+4*(x-1)
```

Input :

```
f:=unapply(integrate(log(t),t,1,x),x)
```

Output :

```
(x)->x*log(x)-x+1
```

Input :

```
f:=unapply(integrate(log(t),t,1,x),x)
f(x)
```

Output :

```
x*log(x)-x+1
```

Remark Suppose that f is a function of 2 variables $f : (x, w) \to f(x, w)$, and that g is the function defined by $g : w \to h_w$ where h_w is the function defined by $h_w(x) = f(x, w)$.
`unapply` is also used to define g with `Xcas`.
Input :

```
f(x,w):=2*x+w
```

```
g(w):=unapply(f(x,w),x)
```

```
g(3)
```

Output :

```
x->2·x+3
```

5.15.3 Top and leaves of an expression : `sommet feuille op`

An operator is an infixed function : for example '+' is an operator and 'sin' is a function.
An expression can be represented by a tree. The top of the tree is either an operator, or a function and the leaves of the tree are the arguments of the operator or of the function (see also 5.38.11).
The instruction `sommet` (resp. `feuille` (or `op`)) returns the top (resp. the list of the leaves) of an expression.
Input :

```
sommet(sin(x+2))
```

Output :

```
'sin'
```

Input :

```
sommet(x+2*y)
```

Output :

```
'+'
```

Input :

```
feuille(sin(x+2))
```

or :

```
op(sin(x+2))
```

Output :

```
x+2
```

Input :

```
feuille(x+2*y)
```

or :

```
op(x+2*y)
```

Output :

```
(x,2*y)
```

Remark
Suppose that a function is defined by a program, for example let us define the `pgcd` function :

```
pgcd(a,b):={local r; while (b!=0)
{r:=irem(a,b);a:=b;b:=r;} return a;}
```

Then input :

```
sommet(pgcd)
```

Output :

```
'program'
```

Then input :

```
feuille(pgcd)[0]
```

Output :

```
(a,b)
```

Then input :

```
feuille(pgcd)[1]
```

Output :

```
(0,0) or (15,25) if the last input was pgcd(15,25)
```

Then input :

```
feuille(pgcd)[2]
```

Output :

```
The body of the program : {local r;....return(a);}
```

5.16 Functions

5.16.1 Context-dependent functions.

Operators + and −

+ (resp. −) is an infixed function and `'+'` (resp. `'-'`) is a prefixed function. The result depends on the nature of its arguments.
Examples with + (all examples except the last one work also with − instead of +) :

- input (1,2)+(3,4) or (1,2,3)+4 or 1+2+3+4 or '+'(1,2,3,4), output 10,
- input 1+i+2+3*i or '+'(1,i,2,3*i), output 3+4*i,
- input [1,2,3]+[4,1] or [1,2,3]+[4,1,0] or '+'([1,2,3],[4,1]), output [5,3,3],

- input [1,2]+[3,4] or '+'([1,2],[3,4]), output [4,6],
- input [[1,2],[3,4]]+ [[1,2],[3,4]], output [[2,4],[6,8]],
- input [1,2,3]+4 or '+'([1,2,3],4), output poly1[1,2,7],
- input [1,2,3]+(4,1) or '+'([1,2,3],4,1), output poly1[1,2,8],
- input "Hel"+"lo" or '+'("Hel","lo"), output "Hello".

Operator $\star$

$\star$ is an infixed function and `'*'` is a prefixed function. The result depends on the nature of its arguments.
Examples with $\star$:

- input (1,2)*(3,4) or (1,2,3)*4 or 1*2*3*4 or '*'(1,2,3,4), output 24,
- input 1*i*2*3*i or '*'(1,i,2,3*i), output -6,
- input [10,2,3]*[4,1] or [10,2,3]*[4,1,0] or '*'([10,2,3],[4,1]), output 42 (scalar product),
- input [1,2]*[3,4] or '*'([1,2],[3,4]), output 11 (scalar product),
- input [[1,2],[3,4]]* [[1,2],[3,4]], output [[7,10],[15,22]],
- input [1,2,3]*4 or '*'([1,2,3],4), output [4,8,12],
- input [1,2,3]*(4,2) or '*'([1,2,3],4,2) or [1,2,3]*8, output [8,16,24],
- input (1,2)+i*(2,3) or 1+2+i*2*3, output 3+6*i.

Operator /

/ is an infixed function and `'/'` is a prefixed function. The result depends of the nature of its arguments.
Examples with / :

- input [10,2,3]/[4,1], output invalid dim
- input [1,2]/[3,4] or '/'([1,2],[3,4]), output [1/3,1/2],
- input 1/[[1,2],[3,4]] or '/'(1,[[1,2],[3,4]], output [[-2,1],[3/2,(-1)/2]],
- input [[1,2],[3,4]]*1/ [[1,2],[3,4]], output [[1,0],[0,1]],
- input [[1,2],[3,4]]/ [[1,2],[3,4]], output [[1,1],[1,1]] (division term by term),

5.16.2 Usual functions

- max takes as argument two real numbers and returns their maximum,
- min takes as argument two real numbers and returns their minimum,
- abs takes as argument a complex number and returns the modulus of the complex parameter (the absolute value if the complex is real),
- sign takes as argument a real number and returns its sign (+1 if it is positive, 0 if it is null, and -1 if it is negative),
- floor (or iPart) takes as argument a real number r, and returns the largest integer $\leq r$,
- round takes as argument a real number and returns its nearest integer,
- ceil or ceiling takes as argument a real number and returns the smallest integer $\geq r$
- frac (or fPart) takes as argument a real number and returns its fractional part,
- trunc takes as argument a real number and returns the integer equal to the real without its fractional part,
- id is the identity function,
- sq is the square function,
- sqrt is the squareroot function,
- exp is the exponential function,
- log or ln is the natural logarithm function,
- log10 is the base-10 logarithm function,
- logb is the logarithm function where the second argument is the base of the logarithm: logb(7,10)=log10(7)=log(7)/log(10),
- sin (resp. cos, tan) is the sinus function, cosinus function, tangent function,
- cot, sec, csc are the cotangent, secant, cosecant function
- asin (or arcsin), acos (or arccos), atan (or arctan), acot, asec, acsc are the inverse trigonometric functions (see section 5.21.1 for more info on trigonometric functions)
- sinh (resp. cosh, tanh) is the hyperbolic sinus function, cosinus function, tangent function,
- asinh or arcsinh (resp. acosh or arccosh, atanh or arctanh) is the inverse function of sinh (resp. cosh, tanh)

5.16.3 Defining algebraic functions

Defining a function from $\mathbb{R}^p$ to $\mathbb{R}$

For $p = 1$, e.g. for $f\ :\ (x) \to x * \sin(x)$, input :

```
f(x):=x*sin(x)
```

or :

```
f:=x->x*sin(x)
```

Output :

```
(x)->x*sin(x)
```

If $p > 1$, e.g. for $f\ :\ (x, y) \to x * \sin(y)$, input :

```
f(x,y):=x*sin(y)
```

or :

```
f:=(x,y)->x*sin(y)
```

Output :

```
(x,y)->x*sin(y)
```

Warning !!! the expression after `->` is not evaluated. You should use `unapply` if you expect the second member to be evaluated before the function is defined.

Defining a function from $\mathbb{R}^p$ to $\mathbb{R}^q$

For example:

- To define the function $h\ :\ (x, y) \to (x * \cos(y), x * \sin(y))$.
  Input :

  ```
  h(x,y):=(x*cos(y),x*sin(y))
  ```

  Output :

  ```
  (x,y)->{
  x*cos(y),x*sin(y);
  }"
  ```

- To define the function $h\ :\ (x, y) \to [x * \cos(y), x * \sin(y)]$.
  Input :

  ```
  h(x,y):=[x*cos(y),x*sin(y)];
  ```

  or :

  ```
  h:=(x,y)->[x*cos(y),x*sin(y)];
  ```

or :

```
h(x,y):={[x*cos(y),x*sin(y)]};
```

or :

```
h:=(x,y)->return[x*cos(y),x*sin(y)];
```

or :

```
h(x,y):={return [x*cos(y),x*sin(y)];}
```

Output :

```
(x,y)->{return([x*cos(y),x*sin(y)]);}
```

Warning !!! The expression after `->` is not evaluated.

Defining families of function from $\mathbb{R}^{p-1}$ to $\mathbb{R}^q$ using a function from $\mathbb{R}^p$ to $\mathbb{R}^q$

Suppose that the function $f : (x, y) \rightarrow f(x, y)$ is defined, and we want to define a family of functions $g(t)$ such that $g(t)(y) := f(t, y)$ (i.e. t is viewed as a parameter). Since the expression after `->` (or `:=`) is not evaluated, we should not define $g(t)$ by `g(t):=y->f(t,y)`, we have to use the `unapply` command.

For example, assuming that $f : (x, y) \rightarrow x \sin(y)$ and $g(t) : y \rightarrow f(t, y)$, input :

```
f(x,y):=x*sin(y);g(t):=unapply(f(t,y),y)
```

Output :

```
((x,y)->x*sin(y), (t)->unapply(f(t,y),y))
```

Input :

```
g(2)
```

Output :

```
y->2· sin(y)
```

Input :

```
g(2)(1)
```

Output :

```
2· sin(1)
```

Next example, suppose that the function $h : (x, y) \rightarrow [x * \cos(y), x * \sin(y)]$ is defined, and we want to define the family of functions $k(t)$ having t as parameter such that $k(t)(y) := h(t, y)$. To define the function $h(x, y)$, input :

```
h(x,y):=(x*cos(y),x*sin(y))
```

To define properly the function $k(t)$, input :

```
k(t):=unapply(h(x,t),x)
```

Output :

```
(t)->unapply(h(x,t),x)
```

Input :

```
k(2)
```

Output :

```
(x)->(x*cos(2),x*sin(2))
```

Input :

```
k(2)(1)
```

Output :

```
(2*cos(1),2*sin(1))
```

5.16.4 Composition of two functions: `@`

With `Xcas`, the composition of functions is done with the infixed operator `@`.
Input :

```
(sq@sin+id)(x)
```

Output :

```
(sin(x))^2+x
```

Input :

```
(sin@sin)(pi/2)
```

Output :

```
sin(1)
```

5.16.5 Repeated function composition: `@@`

With `Xcas`, the repeated composition of a function with itself $n \in \mathbb{N}$ times is done with the infixed operator `@@`.
Input :

```
(sin@@3)(x)
```

Output :

```
sin(sin(sin(x)))
```

Input :

```
(sin@@2)(pi/2)
```

Output :

```
sin(1)
```

5.16.6 Define a function with the history : as_function_of

If an entry defines the variable a and if a later entry defines the variable b (supposed to be dependent on a), then c:=as_function_of(b,a) will define a function c such that c(a)=b.
Input :

```
a:=sin(x)
```

Output :

```
sin(x)
```

Input :

```
b:=sqrt(1+a^2)
```

Output :

```
sqrt(1+sin(x)^2)
```

Input :

```
c:=as_function_of(b,a)
```

Output :

```
(a)->
{ local NULL;
return(sqrt(1+a^2));
}
```

Input :

```
c(x)
```

Output :

```
sqrt(1+x^2)
```

Input :

```
a:=2
```

Output :

```
2
```

Input :

```
b:=1+a^2
```

Output :

```
5
```

Input :

```
c:=as_function_of(b,a)
```

Output :

```
(a)->
{ local NULL;
return(sqrt(1+a^2));
}
```

Input :

```
c(x)
```

Output :

```
1+x^2
```

Warning !!
If the variable b has been assigned several times, the first assignment of b following the last assignment of a will be used. Moreover, the order used is the order of validation of the commandlines, which may not be reflected by the Xcas interface if you reused previous commandlines.
Input for example :
`a:=2` then
`b:=2*a+1` then
`b:=3*a+2` then
`c:=as_function_of(b,a)`
Output :

```
(a)-> {local NULL; return(2*a+1);}
```

i.e. `c(x)` is equal to `2*x+1`.
But, input :
`a:=2` then
`b:=2*a+1` then
`a:=2` then
`b:=3*a+2` then
`c:=as_function_of(b,a)`
Output :

```
(a)-> {local NULL; return(3*a+2);}
```

i.e. `c(x)` is equal to `3*x+2`.
Hence the line where a is defined must be reevaluated before the good definition of b.

5.17 Derivation and applications.

5.17.1 Functional derivative : function_diff

`function_diff` takes a function as argument.
`function_diff` returns the derivative function of this function.
Input :

```
function_diff(sin)
```

Output :

```
(' x')->cos(' x')
```

Input :

```
function_diff(sin)(x)
```

Output :

```
cos(x)
```

Input :

```
f(x):=x^2+x*cos(x)
```

```
function_diff(f)
```

Output :

```
(' x')->2*' x'+cos(' x')+' x'*(-(sin(' x')))
```

Input :

```
function_diff(f)(x)
```

Output :

```
cos(x)+x*(-(sin(x)))+2*x
```

To define the function g as f', input :

```
g:=function_diff(f)
```

The `function_diff` instruction has the same effect as using the expression derivative in conjunction with `unapply` :

```
g:=unapply(diff(f(x),x),x)
```

```
g(x)
```

Output :

```
cos(x)+x*(-(sin(x)))+2*x
```

Warning !!!
In `Maple` mode, for compatibility, `D` may be used in place of `function_diff`. For this reason, it is impossible to assign a variable named `D` in `Maple` mode (hence you can not name a geometric object `D`).

5.17.2 Length of an arc : arcLen

arcLen takes four arguments : an expression ex (resp. a list of two expressions $[ex1, ex2]$), the name of a parameter and two values a and b of this parameter.
arcLen computes the length of the curve define by the equation $y = f(x) = ex$ (resp. by $x = ex1, y = ex2$) when the parameter values varies from a to b, using the formula arcLen(f(x),x,a,b)=
integrate(sqrt(diff(f(x),x)^2+1),x,a,b)
or
integrate(sqrt(diff(x(t),t)^2+diff(y(t),t)^2),t,a,b).

Examples

- Compute the length of the parabola $y = x^2$ from $x = 0$ to $x = 1$.
  Input :

  ```
  arcLen(x^2,x,0,1)
  ```

  or

  ```
  arcLen([t,t^2],t,0,1)
  ```

  Output :

  ```
  -1/4*log(sqrt(5)-2)-(-(sqrt(5)))/2
  ```

- Compute the length of the curve $y = \cosh(x)$ from $x = 0$ to $x = \ln(2)$.
  Input :

  ```
  arcLen(cosh(x),x,0,log(2))
  ```

  Output :

  ```
  3/4
  ```

- Compute the length of the circle $x = \cos(t), y = \sin(t)$ from $t = 0$ to $t = 2 * \pi$.
  Input :

  ```
  arcLen([cos(t),sin(t)],t,0,2*pi)
  ```

  Output :

  ```
  2*pi
  ```

5.17.3 Maximum and minimum of an expression: `fMax fMin`

`fMax` and `fMin` take one or two arguments : an expression of a variable and the name of this variable (by default `x`).
`fMax` returns the abscissa of a maximum of the expression.
`fMin` returns the abscissa of a minimum of the expression.
Input :

```
fMax(sin(x),x)
```

Or :

```
fMax(sin(x))
```

Or :

```
fMax(sin(y),y)
```

Output :

```
pi/2
```

Input :

```
fMin(sin(x),x)
```

Or :

```
fMin(sin(x))
```

Or :

```
fMin(sin(y),y)
```

Output :

```
-pi/2
```

Input :

```
fMin(sin(x)^2,x)
```

Output :

```
0
```

`fMax` and `fMin` can also compute the maximum resp. minimum of a nonlinear multivariate expression subject to a set of nonlinear equality and/or inequality constraints. Both functions in such cases take four to six arguments:

- objective function (an expression)
- list of constraints (equalities and inequalities)
- list of problem variables
- initial guess (must be a list of nonzero reals representing a feasible point)

- precision (optional), if not given the default epsilon value is used
- maximum number of iterations (optional)

The objective function does not need to be differentiable. Both `fMin` and `fMax` return the optimal solution as a vector. Note that the actual optimal value of the objective is not returned.

Although the initial point is required to be feasible, the algorithm will sometimes succeed even if it is infeasible. Note that the initial value of a variable must not be zero.

For example, input :

```
fMin((x-5)^2+y^2-25,[y>=x^2],[x,y],[1,1])
```

Output :

```
[1.2347728624961,1.5246640219568]
```

Input :

```
fMax((x-2)^2+(y-1)^2,[-.25x^2-y^2+1>=0,x-2y+1=0],
                    [x,y],[.5,.75])
```

Output :

```
[-1.82287565553,-0.411437827766]
```

5.17.4 Table of values and graph : `tablefunc` and `plotfunc`

`tablefunc` is a special command that should be run from inside the spreadsheet. It returns the evaluation of an expression ex depending on a variable x for $x = x_0,\ x_0 + h, ...$:

`tablefunc(ex,x,x_0,h)` or `tablefunc(ex,x)`

In the latter case, the default value for x_0 is the default minimum value of x from the graphic configuration and the default value for the step h is 0.1 times the difference between the default maximum and minimum values of x (from the graphic configuration).
Example: type `Alt+t` to open a spreadsheet if none are open. Then select a cell of the spreadsheet (for example `C0`) and to get the table of `"sinus"`, input in the command line of the spreadsheet :

```
tablefunc(sin(x),x)
```

This will fill two columns with the numeric value of `x` and `sin(x)` :

- in the first column the variable `x`, the value of the step `h` (1.0), the minimum value of x (-5.0), then a formula, for example `=C2+C$1`, and the remaining rows of the column is filled by pasting this formula.
- in the next column the function `sin(x)`, the word "Tablefunc", a formula, for example `=evalf(subst(D$0,C$0,C2))`, and the remaining rows of the column are filled by pasting this formula.

Hence the values of `sin(x)` are on the same rows as the values of `x`. Note that the step and begin value and the expression may be easily changed by modifying the correspondent cell.

The graphic representation may be plotted with the `plotfunc` command (see 6.2.1).

5.17.5 Derivative and partial derivative

`diff` or `derive` may have one or two arguments to compute a first order derivative (or first order partial derivative) of an expression or of a list of expressions, or several arguments to compute the n-th partial derivative of an expression or list of expressions.

Derivative and first order partial derivative : `diff derive deriver`

`diff` (or `derive`) takes two arguments : an expression and a variable (resp. a vector of variable names) (see several variable functions in 5.52). If only one argument is provided, the derivative is taken with respect to `x`
`diff` (or `derive`) returns the derivative (resp. a vector of derivatives) of the expression with respect to the variable (resp. with respect to each variable) given as second argument.
Examples :

- Compute :
$$\frac{\partial(xy^2z^3+xyz)}{\partial z}$$
Input :
```
diff(x*y^2*z^3+x*y*z,z)
```
Output :
```
x*y^2*3*z^2+x*y
```
- Compute the 3 first order partial derivatives of $x*y^2*z^3+x*y*z$.
Input :
```
diff(x*y^2*z^3+x*y,[x,y,z])
```
Output :
```
[y^2*z^3+y*z, x*2*y*z^3+x*z, x*y^2*3*z^2+x*y]
```

Derivative and n-th order partial derivative : `diff derive deriver`

`derive` (or `diff`) may take more than two arguments : an expression and the names of the derivation variables (each variable may be followed by $$n$ to indicate the number n of derivations).
`diff` returns the partial derivative of the expression with respect to the variables given after the first argument.

The notation $ is useful if you want to derive k times with respect to the same variable, instead of entering k times the same variable name, one enters the variable name followed by `$k`, for example `x$3` instead of `(x,x,x)`. Each variable may be followed by a $, for example `diff(exp(x*y),x$3,y$2,z)` is the same as `diff(exp(x*y),x,x,x,y,y,z)`

Examples

- Compute :

$$\frac{\partial^2(xy^2z^3+xyz)}{\partial x\partial z}$$

Input :

```
diff(x*y^2*z^3+x*y*z,x,z)
```

Output :

```
y^2*3*z^2+y
```

- Compute :

$$\frac{\partial^3(xy^2z^3+xyz)}{\partial x\partial^2 z}$$

Input :

```
diff(x*y^2*z^3+x*y*z,x,z,z)
```

or :

```
diff(x*y^2*z^3+x*y*z,x,z$2)
```

Output :

```
y^2*3*2*z
```

- Compute the third derivative of :

$$\frac{1}{x^2+2}$$

Input :

```
normal(diff((1)/(x^2+2),x,x,x))
```

or :

```
normal(diff((1)/(x^2+2),x$3))
```

Output :

```
(-24*x^3+48*x)/(x^8+8*x^6+24*x^4+32*x^2+16)
```

Remark

- Note the difference between `diff(f,x,y)` and `diff(f,[x,y])` :
  `diff`(f,x,y) returns $\frac{\partial^2(f)}{\partial x \partial y}$ and
  `diff`$(f,[x,y])$ returns $[\frac{\partial(f)}{\partial x}, \frac{\partial(f)}{\partial y}]$
- Never define a derivative function with `f1(x):=diff(f(x),x)`. Indeed, `x` would mean two different things Xcas is unable to deal with: the variable name to define the f_1 function and the differentiation variable. The right way to define a derivative is either with `function_diff` or:

  ```
  f1:=unapply(diff(f(x),x),x)
  ```

5.18 Integration

5.18.1 Antiderivative and definite integral : `integrate int Int`

`integrate` (or `int`) computes a primitive or a definite integral. A difference between the two commands is that if you input `quest()` just after the evaluation of `integrate`, the answer is written with the $\int$ symbol.

`integrate` (or `int` or `Int`) takes one, two or four arguments.

- with one or two arguments
  an expression or an expression and the name of a variable (by default `x`), `integrate` (or `int`) returns a primitive of the expression with respect to the variable given as second argument.
  Input :

  ```
  integrate(x^2)
  ```

  Output :

  ```
  x^3/3
  ```

  Input :

  ```
  integrate(t^2,t)
  ```

  Output :

  ```
  t^3/3
  ```

- with four arguments :
  an expression, a name of a variable and the bounds of the definite integral, `integrate` (or `int`) returns the exact value of the definite integral if the computation was successful or an unevaluated integral otherwise.
  Input :

```
integrate(x^2,x,1,2)
```

Output :

```
7/3
```

Input :

```
integrate(1/(sin(x)+2),x,0,2*pi)
```

Output after simplification (with the `simplify` command) :

```
2*pi*sqrt(3)/3
```

`Int` is the inert form of `integrate`, it prevents evaluation for example to avoid a symbolic computation that might not be successful if you just want a numeric evaluation.
Input :

```
evalf(Int(exp(x^2),x,0,1))
```

or :

```
evalf(int(exp(x^2),x,0,1))
```

Output :

```
1.46265174591
```

Exercise 1
Let

$$f(x) = \frac{x}{x^2-1} + \ln(\frac{x+1}{x-1})$$

Find a primitive of f.
Input :

```
int(x/(x^2-1)+ln((x+1)/(x-1)))
```

Output :

```
x*log((x+1)/(x-1))+log(x^2-1)+1/2*log(2*x^2/2-1)
```

Or define the function `f`, input :

```
f(x):=x/(x^2-1)+ln((x+1)/(x-1))
```

then input :

```
int(f(x))
```

Output of course the same result.
Warning
For `Xcas`, `log` is the natural logarithm (like `ln`), as `log10` is 10-basis logarithm

Exercise 2
Compute :

$$\int \frac{2}{x^6 + 2 \cdot x^4 + x^2}\, dx$$

Input :

```
int(2/(x^6+2*x^4+x^2))
```

Output :

```
2*((3*x^2+2)/(-(2*(x^3+x)))+-3/2*atan(x))
```

Exercise 3
Compute :

$$\int \frac{1}{\sin(x) + \sin(2 \cdot x)}\, dx$$

Input :

```
integrate(1/(sin(x)+sin(2*x )))
```

Output :

```
(1/-3*log((tan(x/2))^2-3)+1/12*log((tan(x/2))^2))*2
```

5.18.2 Discrete summation: `sum`

`sum` takes two or four arguments :

- four arguments
  an expression, the name of the variable (for example `n`), and the bounds (for example `a` and `b`).
  `sum` returns the discrete sum of this expression with respect to the variable from a to b.
  Input :

  ```
  sum(1,k,-2,n)
  ```

  Output :

  ```
  n+1+2
  ```

  Input :

  ```
  normal(sum(2*k-1,k,1,n))
  ```

  Output :

  ```
  n^2
  ```

Input :

```
sum(1/(n^2),n,1,10)
```

Output :

```
1968329/1270080
```

Input :

```
sum(1/(n^2),n,1,+(infinity))
```

Output :

```
pi^2/6
```

Input :

```
sum(1/(n^3-n),n,2,10)
```

Output :

```
27/110
```

Input :

```
sum(1/(n^3-n),n,1,+(infinity))
```

Output :

```
1/4
```

This result comes from the decomposition of $1/(n\char`^3-n)$.
Input :

```
partfrac(1/(n^3-n))
```

Output :

```
1/(2*(n+1))-1/n+1/(2*(n-1))
```

Hence :
$$\sum_{n=2}^{N}-\frac{1}{n}=-\sum_{n=1}^{N-1}\frac{1}{n+1}=-\frac{1}{2}-\sum_{n=2}^{N-2}\frac{1}{n+1}-\frac{1}{N}$$
$$\frac{1}{2}*\sum_{n=2}^{N}\frac{1}{n-1}=\frac{1}{2}*(\sum_{n=0}^{N-2}\frac{1}{n+1})=\frac{1}{2}*(1+\frac{1}{2}+\sum_{n=2}^{N-2}\frac{1}{n+1})$$
$$\frac{1}{2}*\sum_{n=2}^{N}\frac{1}{n+1}=\frac{1}{2}*(\sum_{n=2}^{N-2}\frac{1}{n+1}+\frac{1}{N}+\frac{1}{N+1})$$
After simplification by $\sum_{n=2}^{N-2}$, it remains :
$$-\frac{1}{2}+\frac{1}{2}*(1+\frac{1}{2})-\frac{1}{N}+\frac{1}{2}*(\frac{1}{N}+\frac{1}{N+1})=\frac{1}{4}-\frac{1}{2N(N+1)}$$
Therefore :

  - for $N = 10$ the sum is equal to : $1/4 - 1/220 = 27/110$
  - for $N = +\infty$ the sum is equal to : $1/4$ because $\frac{1}{2N(N+1)}$ approaches zero when N approaches infinity.

- two arguments
  an expression of one variable (for example f) and the name of this variable (for example x).
  `sum` returns the discrete antiderivative of this expression, i.e. an expression G such that $G_{|x=n+1} - G_{|x=n} = f_{|x=n}$.
  Input :

```
sum(1/(x*(x+1)),x)
```

  Output :

```
-1/x
```

5.18.3 Riemann sum : `sum_riemann`

`sum_riemann` takes two arguments : an expression depending on two variables and the list of the name of these two variables.
`sum_riemann(expression(n,k),[n,k])` returns in the neighborhood of $n = +\infty$ an equivalent of $\sum_{k=1}^{n} expression(n,k)$ (or of $\sum_{k=0}^{n-1} expression(n,k)$ or of $\sum_{k=1}^{n-1} expression(n,k)$) when the sum is looked on as a Riemann sum associated to a continuous function defined on [0,1] or returns `"it is probably not a Riemann sum"` when the no result is found.

Exercise 1
Suppose $S_n = \sum_{k=1}^{n} \frac{k^2}{n^3}$.
Compute $\lim_{n \to +\infty} S_n$.
Input :

```
sum_riemann(k^2/n^3,[n,k])
```

Output :

```
1/3
```

Exercise 2
Suppose $S_n = \sum_{k=1}^{n} \frac{k^3}{n^4}$.
Compute $\lim_{n \to +\infty} S_n$.
Input :

```
sum_riemann(k^3/n^4,[n,k])
```

Output :

```
1/4
```

Exercise 3
Compute $\lim_{n\to+\infty}(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{n+n})$.
Input :

```
sum_riemann(1/(n+k),[n,k])
```

Output :

```
log(2)
```

Exercise 4
Suppose $S_n=\sum_{k=1}^{n}\frac{32n^3}{16n^4-k^4}$.
Compute $\lim_{n\to+\infty}S_n$.
Input :

```
sum_riemann(32*n^3/(16*n^4-k^4),[n,k])
```

Output :

```
2*atan(1/2)+log(3)
```

5.18.4 Integration by parts : `ibpdv` and `ibpu`

`ibpdv`

`ibpdv` is used to search the primitive of an expression written as $u(x).v'(x)$.
`ibpdv` takes two arguments :

- an expression $u(x)*v'(x)$ and $v(x)$ (or a list of two expressions $[F(x), u(x)*v'(x)]$ and $v(x)$),
- or an expression $g(x)$ and 0 (or a list of two expressions $[F(x), g(x)]$ and 0).

`ibpdv` returns :

- if $v(x)\neq 0$, the list $[u(x)v(x), -v(x)u'(x)]$ (or $[F(x){+}u(x)v(x), -v(x)u'(x)]$),
- if the second argument is zero, a primitive of the first argument $g(x)$ (or $F(x)$+a primitive of $g(x)$) :
  hence, `ibpdv(g(x),0)` returns a primitive `G(x)` of `g(x)` or
  `ibpdv([F(x),g(x)],0)` returns `F(x)+G(x)` where `diff(G(x))=g(x)`.

Hence, `ibpdv` returns the terms computed in an integration by parts, with the possibility of doing several `ibpdv`s successively.
When the answer of `ibpdv(u(x)*v'(x),v(x))` is computed, to obtain a primitive of $u(x)v'(x)$, it remains to compute the integral of the second term of this answer and then, to sum this integral with the first term of this answer : to do this, just use `ibpdv` command with the answer as first argument and a new $v(x)$ (or 0 to terminate the integration) as second argument.
Input :

```
ibpdv(ln(x),x)
```

Output :

```
[x ln(x),-1]
```

then

```
ibpdv([x ln(x),-1],0)
```

Output :

```
-x+x ln(x)
```

Remark
When the first argument of `ibpdv` is a list of two elements, `ibpdv` works only on the last element of this list and adds the integrated term to the first element of this list. (therefore it is possible to do several `ibpdv`s successively).
For example :
`ibpdv((log(x))^2,x) = [x*(log(x))^2,-(2*log(x))]`
it remains to integrate `-(2*log(x))`, the input :
`ibpdv(ans(),x)` or input :
`ibpdv([x*(log(x))^2,-(2*log(x))],x)`
Output :
`[x*(log(x))^2+x*(-(2*log(x))),2]`
and it remains to integrate `2`, hence input `ibpdv(ans(),0)` or
`ibpdv([x*(log(x))^2+x*(-(2*log(x))),2],0)`.
Output : `x*(log(x))^2+x*(-(2*log(x)))+2*x`

`ibpu`

`ibpu` is used to search the primitive of an expression written as $u(x).v'(x)$ `ibpu` takes two arguments :

- an expression $u(x){*}v'(x)$ and $u(x)$ (or a list of two expressions $[F(x), u(x){*}v'(x)]$ and $u(x)$),
- an expression $g(x)$ and 0 (or a list of two expressions $[F(x), g(x)]$ and 0).

`ibpu` returns :

- if $u(x) \neq 0$, the list $[u(x) * v(x), -v(x) * u'(x)]$ (or returns the list $[F(x) + u(x) * v(x), -v(x) * u'(x)]$),
- if the second argument is zero, a primitive of the first argument $g(x)$ (or $F(x)$+a primitive of $g(x)$):
  `ibpu(g(x),0)` returns `G(x)` where `diff(G(x))=g(x)` or
  `ibpu([F(x),g(x)],0)` returns `F(x)+G(x)` where `diff(G(x))=g(x)`.

Hence, `ibpu` returns the terms computed in an integration by parts, with the possibility of doing several `ibpu`s successively.
When the answer of `ibpu(u(x)*v'(x),u(x))` is computed, to obtain a primitive of $u(x)v'(x)$, it remains to compute the integral of the second term of this answer and then, to sum this integral with the first term of this answer : to do this, just use `ibpu` command with the answer as first argument and a new $u(x)$ (or 0 to terminate the integration) as second argument.
Input :

```
ibpu(ln(x),ln(x))
```

Output :

```
[x*ln(x),-1]
```

then

```
ibpu([x*ln(x),-1],0)
```

Output :

```
-x+x*ln(x)
```

Remark
When the first argument of `ibpu` is a list of two elements, `ibpu` works only on the last element of this list and adds the integrated term to the first element of this list. (therefore it is possible to do several `ibpu`s successively).
For example :
`ibpu((log(x))^2,log(x)) = [x*(log(x))^2,-(2*log(x))]`
it remains to integrate `-(2*log(x))`, hence input :
`ibpu(ans(),log(x))` or input :
`ibpu([x*(log(x))^2,-(2*log(x))],log(x))`
Output :
`[x*(log(x))^2+x*(-(2*log(x))),2]`
it remains to integrate `2`, hence input :
`ibpu(ans(),0)` or input :
`ibpu([x*(log(x))^2+x*(-(2*log(x))),2],0)`.
Output : `x*(log(x))^2+x*(-(2*log(x)))+2*x`

5.18.5 Change of variables : `subst`

See the `subst` command in the section 5.12.16.

5.19 Limits

5.19.1 Limits : `limit`

`limit` computes the limit of an expression at a finite or infinite point. It is also possible with an optional argument to compute a one-sided limit (1 for the right limit and -1 for the left limit).
`limit` takes three or four arguments :
an expression, the name of a variable (for example `x`), the limit point (for example `a`) and an optional argument, by default `0`, to indicate if the limit is unidirectional. This argument is equal to `-1` for a left limit (`x<a`) or is equal to `1` for a right limit (`x>a`) or is equal to `0` for a limit.
`limit` returns the limit of the expression when the variable (for example `x`) approaches the limit point (for example `a`).
Remark
It is also possible to put `x=a` as argument instead of `x,a`, hence : `limit` takes also as arguments an expression depending of a variable, an equality (variable =value of the limit point) and perhaps 1 or -1 to indicate the direction.
Input :

```
limit(1/x,x,0,-1)
```

or :

```
limit(1/x,x=0,-1)
```

Output :

```
-(infinity)
```

Input :

```
limit(1/x,x,0,1)
```

or :

```
limit(1/x,x=0,1)
```

Output :

```
+(infinity)
```

Input :

```
limit(1/x,x,0,0)
```

or :

```
limit(1/x,x,0)
```

or :

```
limit(1/x,x=0)
```

Output :

```
infinity
```

Hence, `abs(1/x)` approaches $+\infty$ when x approaches 0.

Exercises :

- Find for $n > 2$, the limit when x approaches 0 of :

  $$\frac{n\tan(x) - \tan(nx)}{\sin(nx) - n\sin(x)}$$

  Input :

  ```
  limit((n*tan(x)-tan(n*x))/(sin(n*x)-n*sin(x)),x=0)
  ```

  Output :

  ```
  2
  ```

- Find the limit when x approaches $+\infty$ of :

  $$\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}$$

  Input :

```
limit(sqrt(x+sqrt(x+sqrt(x)))-sqrt(x),x=+infinity)
```

Output :

```
1/2
```

- Find the limit when x approaches 0 of :

$$\frac{\sqrt{1+x+x^2/2}-\exp(x/2)}{(1-\cos(x))\sin(x)}$$

Input :

```
limit((sqrt(1+x+x^2/2)-exp(x/2))/((1-cos(x))*sin(x)),x,0)
```

Output :

```
-1/6
```

Remark
To compute limits, it is better sometimes to quote the first argument.
Input :

```
limit('(2*x-1)*exp(1/(x-1))',x=+infinity)
```

Note that the first argument is quoted, because it is better that this argument is not simplified (i.e. not evaluated).
Output :

```
+(infinity)
```

5.19.2 Integral and limit

Just two examples :

- Find the limit, when a approaches $+\infty$, of :

$$\int_2^a \frac{1}{x^2}\,dx$$

Input :

```
limit(integrate(1/(x^2),x,2,a),a,+(infinity))
```

Output (if `a` is assigned then input `purge(a)`) :

```
1/2
```

- Find the limit, when a approaches $+\infty$, of :

$$\int_2^a (\frac{x}{x^2-1} + \ln(\frac{x+1}{x-1}))\, dx$$

Input :

```
limit(integrate(x/(x^2-1)+log((x+1)/(x-1)),x,2,a),
                    a,+(infinity))
```

Output (if `a` is assigned then input `purge(a)`) :

```
+(infinity)
```

5.20 Rewriting transcendental and trigonometric expressions

5.20.1 Expand a transcendental and trigonometric expression : `texpand` `tExpand`

`texpand` or `tExpand` takes as argument an expression containing transcendental or trigonometric functions.
`texpand` or `tExpand` expands these functions, like simultaneous calling `expexpand`, `lnexpand` and `trigexpand`, for example, $\ln(x^n)$ becomes $n\ln(x)$, $\exp(nx)$ becomes $\exp(x)^n$, $\sin(2x)$ becomes $2\sin(x)\cos(x)$...
Examples :

- 1. Expand $\cos(x+y)$.
  Input :

  ```
  texpand(cos(x+y))
  ```

  Output :

  ```
  cos(x)*cos(y)-sin(x)*sin(y)
  ```

  2. Expand $\cos(3x)$.
  Input :

  ```
  texpand(cos(3*x))
  ```

  Output :

  ```
  4*(cos(x))^ 3-3*cos(x)
  ```

  3. Expand $\frac{\sin(3*x)+\sin(7*x)}{\sin(5*x)}$.
  Input :

  ```
  texpand((sin(3*x)+sin(7*x))/sin(5*x))
  ```

  Output

```
(4*(cos(x))^2-1)*(sin(x)/(16*(cos(x))^4-
12*(cos(x))^2+1))/sin(x)+(64*(cos(x))^6-
80*(cos(x))^4+24*(cos(x))^2-1)*sin(x)/
(16*(cos(x))^4-12*(cos(x))^2+1)/sin(x)
```

Output, after a simplification with `normal(ans())` :

```
4*(cos(x))^2-2
```

- 1. Expand $\exp(x+y)$.
  Input :

  ```
  texpand(exp(x+y))
  ```

  Output :

  ```
  exp(x)*exp(y)
  ```

  2. Expand $\ln(x \times y)$.
  Input :

  ```
  texpand(log(x*y))
  ```

  Output :

  ```
  log(x)+log(y)
  ```

  3. Expand $\ln(x^n)$.
  Input :

  ```
  texpand(ln(x^n))
  ```

  Output :

  ```
  n*ln(x)
  ```

  4. Expand $\ln((e^2)+\exp(2*\ln(2))+\exp(\ln(3)+\ln(2)))$.
  Input :

  ```
  texpand(log(e^2)+exp(2*log(2))+exp(log(3)+log(2)))
  ```

  Output :

  ```
  6+3*2
  ```

  Or input :

  ```
  texpand(log(e^2)+exp(2*log(2)))+
  lncollect(exp(log(3)+log(2)))
  ```

  Output :

  ```
  12
  ```

- Expand $\exp(x+y)+\cos(x+y)+\ln(3x^2)$.
  Input :

  ```
  texpand(exp(x+y)+cos(x+y)+ln(3*x^2))
  ```

  Output :

  ```
  cos(x)*cos(y)-sin(x)*sin(y)+exp(x)*exp(y)+
  ln(3)+2*ln(x)
  ```

5.20.2 Combine terms of the same type : `combine`

`combine` takes two arguments : an expression and the name of a function or class of functions `exp,log,ln, sin,cos,trig`.
Whenever possible, `combine` put together subexpressions corresponding to the second argument:

- `combine(expr,ln)` or `combine(expr,log)` gives the same result as `lncollect(expr)`
- `combine(expr,trig)` or `combine(expr,sin)` or `combine(expr,cos)` gives the same result as `tcollect(expr)`.

Input :

```
combine(exp(x)*exp(y)+sin(x)*cos(x)+ln(x)+ln(y),exp)
```

Output :

```
exp(x+y)+sin(x)*cos(x)+ln(x)+ln(y)
```

Input :

```
combine(exp(x)*exp(y)+sin(x)*cos(x)+ln(x)+ln(y),trig)
```

or

```
combine(exp(x)*exp(y)+sin(x)*cos(x)+ln(x)+ln(y),sin)
```

or

```
combine(exp(x)*exp(y)+sin(x)*cos(x)+ln(x)+ln(y),cos)
```

Output :

```
exp(y)*exp(x)+(sin(2*x))/2+ln(x)+ln(y)
```

Input :

```
combine(exp(x)*exp(y)+sin(x)*cos(x)+ln(x)+ln(y),ln)
```

or

```
combine(exp(x)*exp(y)+sin(x)*cos(x)+ln(x)+ln(y),log)
```

Output :

```
exp(x)*exp(y)+sin(x)*cos(x)+ln(x*y)
```

5.21 Trigonometry

5.21.1 Trigonometric functions

- `sin` is the sine function,
- `cos` is the cosine function,
- `tan` is the tangent function (`tan(x)= sin(x)/cos(x)`),
- `cot` is the cotangent function (`cot(x)= cos(x)/sin(x)`),
- `sec` is the secant function (`sec(x)= 1/cos(x)`),
- `csc` is the cosecant function (`csc(x) = 1/sin(x)`),
- `asin` or `arcsin`, `acos` or `arccos`, `atan` or `arctan`, `acot`, `asec`, `acsc` are the inverse trigonometric functions. The latter are defined by:
  1. `asec(x) = acos(1/x)`,
  2. `acsc(x) = asin(1/x)`,
  3. `acot(x) = atan(1/x)`.

5.21.2 Expand a trigonometric expression : `trigexpand`

`trigexpand` takes as argument an expression containing trigonometric functions.
`trigexpand` expands sums, differences and products by an integer inside the trigonometric functions
Input :

```
trigexpand(cos(x+y))
```

Output :

```
cos(x)*cos(y)-sin(x)*sin(y)
```

5.21.3 Linearize a trigonometric expression : `tlin`

`tlin` takes as argument an expression containing trigonometric functions.
`tlin` linearizes products and integer powers of the trigonometric functions (e.g. in terms of $\sin(n*x)$ and $\cos(n*x)$)
Examples

- Linearize $\cos(x)*\cos(y)$.
  Input :

  ```
  tlin(cos(x)*cos(y))
  ```

  Output :

  ```
  1/2*cos(x-y)+1/2*cos(x+y)
  ```

- Linearize $\cos(x)^3$.
  Input :

  ```
  tlin(cos(x)^3)
  ```

  Output :

  ```
  3/4*cos(x)+1/4*cos(3*x)
  ```

- Linearize $4\cos(x)^2 - 2$.
  Input :

  ```
  tlin(4*cos(x)^2-2)
  ```

  Output :

  ```
  2*cos(2*x)
  ```

5.21.4 Put together sine and cosine of the same angle : `tcollect` `tCollect`

`tcollect` or `tCollect` takes as argument an expression containing trigonometric functions.
`tcollect` first linearizes this expression (e.g. in terms of $\sin(n*x)$ and $\cos(n*x)$), then, puts together sine and cosine of the same angle.
Input :

```
tcollect(sin(x)+cos(x))
```

Output :

```
sqrt(2)*cos(x-pi/4)
```

Input :

```
tcollect(2*sin(x)*cos(x)+cos(2*x))
```

Output :

```
sqrt(2)*cos(2*x-pi/4)
```

5.21.5 Simplify : `simplify`

`simplify` simplifies the expression.
As with all automatic simplifications, do not expect miracles, you will have to use specific rewriting rules if it does not work.
Input :

```
simplify((sin(3*x)+sin(7*x))/sin(5*x))
```

Output :

```
4*(cos(x))^2-2
```

Warning `simplify` is more efficient in `radian` mode (check `radian` in the `cas` configuration or input `angle_radian:=1`).

5.21.6 Simplify trigonometric expressions : trigsimplify

trigsimplify simplifies trigonometric expressions by combining simplify, texpand, tlin, tcollect, trigsin, trigcos and trigtan commands in a certain order.
Input :

```
trigsimplify((sin(x+y)-sin(x-y))/(cos(x+y)+cos(x-y)))
```

Output :

```
tan(4*x/3)
```

Input :

```
trigsimplify(1-1/4*sin(2a)^2-sin(b)^2-cos(a)^4)
```

Output :

```
sin(a)^2-sin(b)^2
```

5.21.7 Transform arccos into arcsin : acos2asin

acos2asin takes as argument an expression containing inverse trigonometric functions.
acos2asin replaces $\arccos(x)$ by $\frac{\pi}{2} - \arcsin(x)$ in this expression.
Input :

```
acos2asin(acos(x)+asin(x))
```

Output after simplification :

```
pi/2
```

5.21.8 Transform arccos into arctan : acos2atan

acos2atan takes as argument an expression containing inverse trigonometric functions.
acos2atan replaces $\arccos(x)$ by $\frac{\pi}{2} - \arctan(\frac{x}{\sqrt{1-x^2}})$ in this expression.
Input :

```
acos2atan(acos(x))
```

Output :

```
pi/2-atan(x/sqrt(1-x^2))
```

5.21.9 Transform arcsin into arccos : asin2acos

asin2acos takes as argument an expression containing inverse trigonometric functions.
asin2acos replaces $\arcsin(x)$ by $\frac{\pi}{2} - \arccos(x)$ in this expression.
Input :

```
asin2acos(acos(x)+asin(x))
```

Output after simplification :

```
pi/2
```

5.21.10 Transform arcsin into arctan : `asin2atan`

`asin2atan` takes as argument an expression containing inverse trigonometric functions.
`asin2atan` replaces $\arcsin(x)$ by $\arctan(\frac{x}{\sqrt{1-x2}})$ in this expression.
Input :

```
asin2atan(asin(x))
```

Output :

```
atan(x/sqrt(1-x^2))
```

5.21.11 Transform arctan into arcsin : `atan2asin`

`atan2asin` takes as argument an expression containing inverse trigonometric functions. `atan2asin` replaces $\arctan(x)$ by $\arcsin(\frac{x}{\sqrt{1+x2}})$ in this expression.
Input :

```
atan2asin(atan(x))
```

Output :

```
asin(x/sqrt(1+x^2))
```

5.21.12 Transform arctan into arccos : `atan2acos`

`atan2acos` takes as argument an expression containing inverse trigonometric functions.
`atan2acos` replaces $\arctan(x)$ by $\frac{\pi}{2} - \arccos(\frac{x}{\sqrt{1+x2}})$ in this expression.
Input :

```
atan2acos(atan(x))
```

Output :

```
pi/2-acos(x/sqrt(1+x^2))
```

5.21.13 Transform complex exponentials into sin and cos : `sincos` `exp2trig`

`sincos` or `exp2trig` takes as argument an expression containing complex exponentials.
`sincos` or `exp2trig` rewrites this expression in terms of sin and cos.
Input :

```
sincos(exp(i*x))
```

Output :

```
cos(x)+(i)*sin(x)
```

Input :

```
exp2trig(exp(-i*x))
```

Output :

```
cos(x)+(i)*(-(sin(x)))
```

Input :

```
simplify(sincos(((i)*(exp((i)*x))^2-i)/(2*exp((i)*x))))
```

or :

```
simplify(exp2trig(((i)*(exp((i)*x))^2-i)/(2*exp((i)*x))))
```

Output :

```
-sin(x)
```

5.21.14 Transform tan(x) into sin(x)/cos(x) : `tan2sincos`

`tan2sincos` takes as argument an expression containing trigonometric functions.
`tan2sincos` replaces $\tan(x)$ by $\frac{\sin(x)}{\cos(x)}$ in this expression.
Input :

```
tan2sincos(tan(2*x))
```

Output :

```
sin(2*x)/cos(2*x)
```

5.21.15 Rewrite tan(x) with sin(2x) and cos(2x) : `tan2sincos2`

`tan2sincos2` takes as argument an expression containing trigonometric functions.
`tan2sincos2` replaces $\tan(x)$ by $\frac{\sin(2x)}{1+\cos(2x)}$ in this expression.
Input :

```
tan2sincos2(tan(x))
```

Output :

```
sin(2*x)/(1+cos(2*x))
```

5.21.16 Rewrite tan(x) with cos(2x) and sin(2x) : `tan2cossin2`

`tan2cossin2` takes as argument an expression containing trigonometric functions.
`tan2cossin2` replaces $\tan(x)$ by $\dfrac{1-\cos(2x)}{\sin(2x)}$, in this expression.
Input :

```
tan2cossin2(tan(x))
```

Output :

```
(1-cos(2*x))/sin(2*x)
```

5.21.17 Rewrite sin, cos, tan in terms of tan(x/2) : `halftan`

`halftan` takes as argument an expression containing trigonometric functions.
`halftan` rewrites $\sin(x),\ \cos(x)$ and $\tan(x)$ in terms of $\tan(\frac{x}{2})$.
Input :

```
halftan(sin(2*x)/(1+cos(2*x)))
```

Output :

```
2*tan(2*x/2)/((tan(2*x/2))^2+1)/
(1+(1-(tan(2*x/2))^2)/((tan(2*x/2))^2+1))
```

Output, after simplification with `normal(ans())` :

```
tan(x)
```

Input :

```
halftan(sin(x)^2+cos(x)^2)
```

Output :

```
(2*tan(x/2)/((tan(x/2))^2+1))^2+
((1-(tan(x/2))^2)/((tan(x/2))^2+1))^2
```

Output, after simplification with `normal(ans())` :

```
1
```

5.21.18 Rewrite trigonometric functions as function of tan(x/2) and hyperbolic functions as function of exp(x): `halftan_hyp2exp`

`halftan_hyp2exp` takes as argument a trigonometric and hyperbolic expression.
`halftan_hyp2exp` rewrites $\sin(x),\ \cos(x), \tan(x)$ in terms of $\tan(\frac{x}{2})$ and $\sinh(x),\ \cosh(x), \tanh(x)$ in terms of $\exp(x)$.
Input :

```
halftan_hyp2exp(tan(x)+tanh(x))
```

Output :

```
(2*tan(x/2))/((1-(tan(x/2))^2))+(((exp(x))^2-1))/
(((exp(x))^2+1))
```

Input :

```
halftan_hyp2exp(sin(x)^2+cos(x)^2-sinh(x)^2+cosh(x)^2)
```

Output, after simplification with `normal(ans())` :

```
2
```

5.21.19 Transform inverse trigonometric functions into logarithms : `atrig2ln`

`atrig2ln` takes as argument an expression containing inverse trigonometric functions.
`atrig2ln` rewrites these functions with complex logarithms.
Input :

```
atrig2ln(asin(x))
```

Output :

```
i*log(x+sqrt(x^2-1))+pi/2
```

5.21.20 Transform trigonometric functions into complex exponentials : `trig2exp`

`trig2exp` takes as argument an expression containing trigonometric functions.
`trig2exp` rewrites the trigonometric functions with complex exponentials (WITHOUT linearization).
Input :

```
trig2exp(tan(x))
```

Output :

```
((exp((i)*x))^2-1)/((i)*((exp((i)*x))^2+1))
```

Input :

```
trig2exp(sin(x))
```

Output :

```
(exp((i)*x)-1/(exp((i)*x)))/(2*i)
```

5.21.21 Simplify and express preferentially with sine : `trigsin`

`trigsin` takes as argument an expression containing trigonometric functions.
`trigsin` simplify this expression with the formula :
$\sin(x)^2+\cos(x)^2=1$, $\tan(x)=\frac{\sin(x)}{\cos(x)}$ and tries to rewrite the expression only with sine.
Input :

```
trigsin(sin(x)^4+cos(x)^2+1)
```

Output :

```
sin(x)^4-sin(x)^2+2
```

5.21.22 Simplify and express preferentially with cosine : `trigcos`

`trigcos` takes as argument an expression containing trigonometric functions.
`trigcos` simplifies this expression with the formula :
$\sin(x)^2+\cos(x)^2=1$, $\tan(x)=\frac{\sin(x)}{\cos(x)}$ and tries to rewrite the expression only with cosine.
Input :

```
trigcos(sin(x)^4+cos(x)^2+1)
```

Output :

```
cos(x)^4-cos(x)^2+2
```

5.21.23 Simplify and express preferentially with tangents : `trigtan`

`trigtan` takes as argument an expression containing trigonometric functions.
`trigtan` simplifies this expression with the formula :
$\sin(x)^2+\cos(x)^2=1$, $\tan(x)=\frac{\sin(x)}{\cos(x)}$ and tries to rewrite the expression only with tangents.
Input :

```
trigtan(sin(x)^4+cos(x)^2+1)
```

Output :

```
((tan(x))^2/(1+(tan(x))^2))^2+1/(1+(tan(x)^2)+1
```

Output, after simplification with `normal` :

```
(2*tan(x)^4+3*tan(x)^2+2)/(tan(x)^4+2*tan(x))^2+1)
```

5.21.24 Rewrite an expression with different options : `convert` `convertir`

`convert` takes two arguments an expression and an option.
`convert` rewrites this expression applying rules depending on the option. Valid options are :

- `sin` converts an expression like `trigsin`.
- `cos` converts an expression like `trigcos`.
- `sincos` converts an expression like `sincos`.
- `trig` converts an expression like `sincos`.
- `tan` converts an expression like `halftan`.
- `exp` converts an expression like `trig2exp`.
- `ln` converts an expression like `trig2exp`.
- `expln` converts an expression like `trig2exp`.
- `string` converts an expression into a string.
- `matrix` converts a list of lists into a matrix.
- `polynom` converts a Taylor series into a polynomial by removing the remainder (cf 5.25.22).
- `parfrac` or `partfrac` or `fullparfrac` converts a rational fraction into its partial fraction decomposition (5.29.9).

`convert` can also :

- convert units, for example `convert(1000_g,_kg)=1.0_kg` (cf 8.1.4).
- write a real as a continued fraction : `convert(a,confrac,'fc')` writes `a` as a continued fraction stored in `fc`. Do not forget to quote the last argument if it was assigned.
  For example, `convert(1.2,confrac,'fc')=[1,5]` and `fc` contains the continued fraction equal to 1.2 (cf 5.8.7).
- transform an integer into the list of its digits in a base, beginning with the units digit (and reciprocally)
  - `convert(n,base,b)` transforms the integer `n` into the list of its digits in base `b` beginning with the units digit.
    For example, `convert(123,base,10)=[3,2,1]` and reciprocally
  - `convert(l,base,b)` transforms the list `l` into the integer `n` which has `l` as list of its digits in base `b` beginning with the units digit.
    For example, `convert([3,2,1],base,10)=123` (cf 5.5).

5.22 Fourier transformation

5.22.1 Fourier coefficients : `fourier_an` and `fourier_bn` or `fourier_cn`

Let f be a T-periodic continuous functions on $\mathbb{R}$ except maybe at a finite number of points. One can prove that if f is continuous at x, then;

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(\frac{2\pi n x}{T}) + b_n \sin(\frac{2\pi n x}{T}) \\ &= \sum_{n=-\infty}^{+\infty} c_n e^{\frac{2i\pi n x}{T}} \end{aligned}$$

where the coefficients $a_n,\ b_n, n \in N$, (or $c_n, n \in Z$) are the Fourier coefficients of f. The commands`fourier_an` and `fourier_bn` or `fourier_cn` compute these coefficients.

`fourier_an`

`fourier_an` takes four or five arguments : an expression *expr* depending on a variable, the name of this variable (for example x), the period T, an integer n and a real a (by default $a = 0$).
`fourier_an(expr,x,T,n,a)` returns the Fourier coefficient a_n of a function f of variable x defined on $[a, a+T)$ by $f(x) = expr$ and such that f is periodic of period T:

$$a_n = \frac{2}{T} \int_a^{a+T} f(x) \cos(\frac{2\pi n x}{T}) dx$$

To simplify the computations, one should input `assume(n,integer)` before calling `fourier_an` to specify that n is an integer.
Example Let the function f, of period $T = 2$, defined on $[-1, 1)$ by $f(x) = x^2$.
Input, to have the coefficient a_0 :

```
fourier_an(x^2,x,2,0,-1)
```

Output :

```
1/3
```

Input, to have the coefficient a_n ($n \neq 0$) :

```
assume(n,integer);fourier_an(x^2,x,2,n,-1)
```

Output :

```
4*(-1)^n/(pi^2*n^2)
```

`fourier_bn`

`fourier_bn` takes four or five arguments : an expression *expr* depending on a variable, the name of this variable (for example x), the period T, an integer n and a real a (by default $a = 0$).

`fourier_bn(expr,x,T,n,a)` returns the Fourier coefficient b_n of a function f of variable x defined on $[a, a+T)$ by $f(x) = expr$ and periodic of period T:

$$b_n = \frac{2}{T}\int_a^{a+T} f(x)\sin(\frac{2\pi nx}{T})dx$$

To simplify the computations, one should input `assume(n,integer)` before calling `fourier_bn` to specify that n is an integer.

Examples

- Let the function f, of period $T = 2$, defined on $[-1, 1)$ by $f(x) = x^2$.
  Input, to have the coefficient b_n $(n \neq 0)$:

  ```
  assume(n,integer);fourier_bn(x^2,x,2,n,-1)
  ```

  Output :

  ```
  0
  ```

- Let the function f, of period $T = 2$, defined on $[-1, 1)$ by $f(x) = x^3$.
  Input, to have the coefficient b_1 :

  ```
  fourier_bn(x^3,x,2,1,-1)
  ```

  Output :

  ```
  (2*pi^2-12)/pi^3
  ```

`fourier_cn`

`fourier_cn` takes four or five arguments : an expression $expr$ depending of a variable, the name of this variable (for example x), the period T, an integer n and a real a (by default $a = 0$).
`fourier_cn(expr,x,T,n,a)` returns the Fourier coefficient c_n of a function f of variable x defined on $[a, a+T)$ by $f(x) = expr$ and periodic of period T:

$$c_n = \frac{1}{T}\int_a^{a+T} f(x)e^{\frac{-2i\pi nx}{T}}dx$$

To simplify the computations, one should input `assume(n,integer)` before calling `fourier_cn` to specify that n is an integer.

Examples

- Find the Fourier coefficients c_n of the periodic function f of period 2 and defined on $[-1, 1)$ by $f(x) = x^2$.
  Input, to have c_0 :

  ```
  fourier_cn(x^2,x,2,0,-1)
  ```

  Output:

```
1/3
```

Input, to have c_n :

```
assume(n,integer)
```

```
fourier_cn(x^2,x,2,n,-1)
```

Output:

```
2*(-1)^n/(pi^2*n^2)
```

- Find the Fourier coefficients c_n of the periodic function f, of period 2, and defined on $[0, 2)$ by $f(x) = x^2$.
  Input, to have c_0 :

  ```
  fourier_cn(x^2,x,2,0)
  ```

  Output:

  ```
  4/3
  ```

  Input, to have c_n :

  ```
  assume(n,integer)
  ```

  ```
  fourier_cn(x^2,x,2,n)
  ```

  Output:

  ```
  ((2*i)*pi*n+2)/(pi^2*n^2)
  ```

- Find the Fourier coefficients c_n of the periodic function f of period 2π and defined on $[0, 2\pi)$ by $f(x) = x^2$.
  Input :

  ```
  assume(n,integer)
  ```

  ```
  fourier_cn(x^2,x,2*pi,n)
  ```

  Output :

  ```
  ((2*i)*pi*n+2)/n^2
  ```

  If you don't specify `assume(n,integer)`, the output will not be simplified :

```
((2*i)*pi^2*n^2*exp((-i)*n*2*pi)+2*pi*n*exp((-i)*n*2*pi)+
        (-i)*exp((-i)*n*2*pi)+i)/(pi*n^3)
```

You might simplify this expression by replacing `exp((-i)*n*2*pi)` by `1`, input :

```
subst(ans(),exp((-i)*n*2*pi)=1)
```

Output :

```
((2*i)*pi^2*n^2+2*pi*n+-i+i)/pi/n^3
```

This expression is then simplified with `normal`, the final output is :

```
((2*i)*pi*n+2)/n^2
```

Hence for $n \neq 0$, $c_n = \dfrac{2in\pi + 2}{n^2}$. As shown in this example, it is better to input `assume(n,integer)` before calling `fourier_cn`.
We must also compute c_n for $n = 0$, input :

```
fourier_cn(x^2,x,2*pi,0)
```

Output :

```
4*pi^2/3
```

Hence for $n = 0$, $c_0 = \dfrac{4\pi^2}{3}$.

Remarks :

- Input `purge(n)` to remove the hypothesis done on n.
- Input `about(n)` or `assume(n)`, to know the hypothesis done on the variable n.

5.22.2 Discrete Fourier Transform

Let N be an integer. The Discrete Fourier Transform (DFT) is a transformation F_N defined on the set of periodic sequences of period N, it depends on a choice of a primitive N-th root of unity ω_N. If the DFT is defined on sequences with complex coefficients, we take:

$$\omega_N = e^{\frac{2i\pi}{N}}$$

If x is a periodic sequence of period N, defined by the vector $x = [x_0, x_1, ...x_{N-1}]$ then $F_N(x) = y$ is a periodic sequence of period N, defined by:

$$(F_{N,\omega_N}(x))_k = y_k = \sum_{j=0}^{N-1} x_j \omega_N^{-k\cdot j}, k = 0..N-1$$

where ω_N is a primitive N-th root of unity. The discrete Fourier transform may be computed faster than by computing each y_k individually, by the Fast Fourier Transform (FFT). `Xcas` implements the FFT algorithm to compute the discrete Fourier transform only if N is a power of 2.

The properties of the Discrete Fourier Transform

The Discrete Fourier Transform F_N is a bijective transformation on periodic sequences such that

$$\begin{aligned} F_{N,\omega_N}^{-1} &= \frac{1}{N} F_{N,\omega_N^{-1}} \\ &= \frac{1}{N} \overline{F_N} \quad \text{on } \mathbb{C} \end{aligned}$$

i.e. :

$$(F_N^{-1}(x))_k = \frac{1}{N} \sum_{j=0}^{N-1} x_j \omega_N^{k \cdot j}$$

Inside `Xcas` the discrete Fourier transform and its inverse are denote by `fft` and `ifft`:

`fft(x)`$=F_N(x)$, `ifft(x)`$=F_N^{-1}(x)$

Definitions
Let x and y be two periodic sequences of period N.

- The Hadamard product (notation $\cdot$) is defined by:

$$(x \cdot y)_k = x_k y_k$$

- the convolution product (notation $*$) is defined by:

$$(x * y)_k = \sum_{j=0}^{N-1} x_j y_{k-j}$$

Properties :

$$\begin{aligned} N * F_N(x \cdot y) &= F_N(x) * F_N(y) \\ F_N(x * y) &= F_N(x) \cdot F_N(y) \end{aligned}$$

Applications

1. Value of a polynomial
Define a polynomial $P(x) = \sum_{j=0}^{N-1} c_j x^j$ by the vector of its coefficients $c := [c_0, c_1, ..c_{N-1}]$, where zeroes may be added so that N is a power of 2.

   - Compute the values of $P(x)$ at

   $$x = a_k = \omega_N^{-k} = \exp(\frac{-2ik\pi}{N}), \quad k = 0..N-1$$

   This is just the discrete Fourier transform of c since

   $$P(a_k) = \sum_{j=0}^{N-1} c_j (\omega_N^{-k})^j = F_N(c)_k$$

   Input, for example :

```
P(x):=x+x^2; w:=i
```

Here the coefficients of P are [0,1,1,0], $N = 4$ and $\omega = \exp(2i\pi/4) = i$.
Input :
`fft([0,1,1,0])`
Output :
`[2,-1-i,0,-1+i]`
hence

- `P(1)=2`,
- `P(-i)=P(w^-1)=-1-i`,
- `P(-1)=P(w^-2)=0`,
- `P(i)=P(w^-3)=-1+i`.

• Compute the values of $P(x)$ at

$$x = b_k = \omega_N^k = \exp(\frac{2ik\pi}{N}), \quad k = 0..N-1$$

This is N times the inverse fourier transform of c since

$$P(a_k) = \sum_{j=0}^{N-1} c_j(\omega_N^k)^j = NF_N^{-1}(c)_k$$

Input, for example :
`P(x):=x+x^2` and `w:=i`
Hence, the coefficients of P are [0,1,1,0], $N = 4$ and $\omega = \exp(2i\pi/4) = i$.
Input :
`4*ifft([0,1,1,0])`
Output :
`[2,-1+i,0,-1-i]`
hence :

- `P(1)=2`,
- `P(i)=P(w^1)=-1+i`,
- `P(-1)=P(w^2)=0`,
- `P(-i)=P(w^3)=-1-i`.

We find of course the same values as above...

2. Trigonometric interpolation
Let f be periodic function of period 2π, assume that $f(2k\pi/N) = f_k$ for $k = 0..(N-1)$. Find a trigonometric polynomial p that interpolates f at $x_k = 2k\pi/N$, that is find $p_j, j = 0..N-1$ such that

$$p(x) = \sum_{j=0}^{N-1} p_j \exp(ijx), \quad p(x_k) = f_k$$

Replacing x_k by its value in $p(x)$ we get:

$$\sum_{j=0}^{N-1} p_j \exp(i\frac{j2k\pi}{N}) = f_k$$

In other words, (f_k) is the inverse DFT of (p_k), hence

$$(p_k) = \frac{1}{N} F_N(\ (f_k)\ )$$

If the function f is real, $p_{-k} = \overline{p}_k$, hence depending whether N is even or odd:

$$\begin{aligned} p(x) &= p_0 + 2\Re(\sum_{k=0}^{\frac{N}{2}-1} p_k \exp(ikx)) + \Re(p_{\frac{N}{2}} \exp(i\frac{Nx}{2})) \\ p(x) &= p_0 + 2\Re(\sum_{k=0}^{\frac{N-1}{2}} p_k \exp(ikx)) \end{aligned}$$

3. Fourier series
   Let f be a periodic function of period 2π, such that

$$f(x_k) = y_k, \quad x_k = \frac{2k\pi}{N}, k = 0..N-1$$

Suppose that the Fourier series of f converges to f (this will be the case if for example f is continuous). If N is large, a good approximation of f will be given by:

$$\sum_{-\frac{N}{2} \leq n < \frac{N}{2}} c_n \exp(inx)$$

Hence we want a numeric approximation of

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) \exp(-int) dt$$

The numeric value of the integral $\int_0^{2\pi} f(t) \exp(-int) dt$ may be computed by the trapezoidal rule (note that the Romberg algorithm would not work here, because the Euler Mac Laurin development has its coefficients equal to zero, since the integrated function is periodic, hence all its derivatives have the same value at 0 and at 2π). If $\tilde{c_n}$ is the numeric value of c_n obtained by the trapezoidal rule, then

$$\tilde{c_n} = \frac{1}{2\pi} \frac{2\pi}{N} \sum_{k=0}^{N-1} y_k \exp(-2i\frac{nk\pi}{N}), \quad -\frac{N}{2} \leq n < \frac{N}{2}$$

Indeed, since $x_k = 2k\pi/N$ and $f(x_k) = y_k$:

$$\begin{aligned} f(x_k) \exp(-inx_k) &= y_k \exp(-2i\frac{nk\pi}{N}), \\ f(0) \exp(0) = f(2\pi) \exp(-2i\frac{nN\pi}{N}) &= y_0 = y_N \end{aligned}$$

Hence :

$$[\tilde{c}_0, ..\tilde{c}_{\frac{N}{2}-1}, \tilde{c}_{\frac{N}{2}+1}, ..c_{N-1}] = \frac{1}{N} F_N([y_0, y_1 ... y_{(N-1)}])$$

since

- if $n \geq 0$, $\tilde{c}_n = y_n$
- if $n < 0$ $\tilde{c}_n = y_{n+N}$
- $\omega_N = \exp(\frac{2i\pi}{N})$, then $\omega_N^n = \omega_N^{n+N}$

Properties

- The coefficients of the trigonometric polynomial that interpolates f at $x = 2k\pi/N$ are
$$p_n = \tilde{c}_n, \quad -\frac{N}{2} \leq n < \frac{N}{2}$$
- If f is a trigonometric polynomial P of degree $m \leq \frac{N}{2}$, then
$$f(t) = P(t) = \sum_{k=-m}^{m-1} c_k \exp(2ik\pi t)$$
the trigonometric polynomial that interpolate $f = P$ is P, the numeric approximation of the coefficients are in fact exact ($\tilde{c}_n = c_n$).
- More generally, we can compute $\tilde{c}_n - c_n$.
Suppose that f is equal to its Fourier series, i.e. that :
$$f(t) = \sum_{m=-\infty}^{+\infty} c_m \exp(2i\pi mt), \qquad \sum_{m=-\infty}^{+\infty} |c_m| < \infty$$
Then :
$$f(x_k) = f(\frac{2k\pi}{N}) = y_k = \sum_{m=-\infty}^{+\infty} c_m \omega_N^{km}, \quad \tilde{c_n} = \frac{1}{N}\sum_{k=0}^{N-1} y_k \omega_N^{-kn}$$
Replace y_k by its value in $\tilde{c_n}$:
$$\tilde{c_n} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{+\infty} c_m \omega_N^{km}\omega_N^{-kn}$$
If $m \neq n \pmod N$, ω_N^{m-n} is an N-th root of unity different from 1, hence:
$$\omega_N^{(m-n)N} = 1, \quad \sum_{k=0}^{N-1} \omega_N^{(m-n)k} = 0$$
Therefore, if $m-n$ is a multiple of N ($m = n+l\cdot N$) then $\sum_{k=0}^{N-1} \omega_N^{k(m-n)} = N$, otherwise $\sum_{k=0}^{N-1} \omega_N^{k(m-n)} = 0$. By reversing the two sums, we get
$$\begin{aligned}\tilde{c_n} &= \frac{1}{N}\sum_{m=-\infty}^{+\infty} c_m \sum_{k=0}^{N-1} \omega_N^{k(m-n)} \\ &= \sum_{l=-\infty}^{+\infty} c_{(n+l\cdot N)} \\ &= ...c_{n-2\cdot N} + c_{n-N} + c_n + c_{n+N} + c_{n+2\cdot N} + ...\end{aligned}$$
Conclusion: if $|n| < N/2$, $\tilde{c_n} - c_n$ is a sum of c_j of large indexes (at least $N/2$ in absolute value), hence is small (depending on the rate of convergence of the Fourier series).

Example Input :

```
f(t):=cos(t)+cos(2*t)
x:=f(2*k*pi/8)$(k=0..7)
```

Then :

```
x={2,sqrt(2)/2,-1,(-sqrt(2)/2,0,(-sqrt(2))/2,-1,sqrt(2)/2}
fft(x)=[0.0,4.0,4.0,0.0,0.0,0.0,4.0,4.0]
```

After a division by $N = 8$, we get

$$c_0 = 0, c_1 = 4.0/8, c_2 = 4.0/8, c_3 = 0.0,$$
$$c_{-4} = 0.0, c_{-3} = 0.0, c_{-2} = 4.0/8, = c_{-1} = 4.0/8$$

Hence $b_k = 0$ and $a_k = c_{-k} + c_k$ is equal to 1 if $k = 1, 2$ and 0 otherwise.

4. Convolution Product
If $P(x) = \sum_{j=0}^{n-1} a_j x^j$ and $Q(x) = \sum_{j=0}^{m-1} b_j x^j$ are given by the vector of their coefficients $a = [a_0, a_1, ..a_{n-1}]$ and $b = [b_0, b_1, ..b_{m-1}]$, we may compute the product of these two polynomials using the DFT. The product of polynomials is the convolution product of the periodic sequence of their coefficients if the period is greater or equal to $(n + m)$. Therefore we complete a (resp. b) with $m + p$ (resp. $n + p$) zeros, where p is chosen such that $N = n + m + p$ is a power of 2. If $a = [a_0, a_1, ..a_{n-1}, 0..0]$ and $b = [b_0, b_1, ..b_{m-1}, 0..0]$, then:

$$P(x)Q(x) = \sum_{j=0}^{n+m-1} (a * b)_j x^j$$

We compute $F_N(a)$, $F_N(b)$, then $ab = F_N^{-1}(F_N(a) \cdot F_N(b))$ using the properties

$$NF_N(x \cdot y) = F_N(x) * F_N(y), \quad F_N(x * y) = F_N(x) \cdot F_N(y)$$

5.22.3 Fast Fourier Transform : `fft`

`fft` takes as argument a list (or a sequence) $[\mathtt{a}_0, ..\mathtt{a}_{\mathtt{N}-1}]$ where `N` is a power of two. `fft` returns the list $[\mathtt{b}_0, ..\mathtt{b}_{\mathtt{N}-1}]$ such that, for `k=0..N-1`

$$\mathtt{fft}([\mathtt{a}_0, ..\mathtt{a}_{\mathtt{N}-1}])[\mathtt{k}] = \mathtt{b}_\mathtt{k} = \sum_{\mathtt{j}=0}^{\mathtt{N}-1} \mathtt{x}_\mathtt{j} \omega_\mathtt{N}^{-\mathtt{k}\cdot\mathtt{j}}$$

where ω_N is a primitive N-th root of the unity.
Input :

```
fft(0,1,1,0)
```

Output :

```
[2.0, -1-i, 0.0, -1+i]
```

5.22.4 Inverse Fast Fourier Transform : ifft

ifft takes as argument a list $[b_0, ..b_{N-1}]$ where N is a power of two.
ifft returns the list $[a_0, ..a_{N-1}]$ such that

$$\texttt{fft}([a_0, ..a_{N-1}]) = [b_0, ..b_{N-1}]$$

Input :

```
ifft([2,-1-i,0,-1+i])
```

Output :

```
[0.0, 1.0, 1.0, 0.0]
```

5.22.5 An exercise with fft

Here are the temperatures T, in Celsius degree, at time t :

t	0	3	6	9	12	15	19	21
T	11	10	17	24	32	26	23	19

What was the temperature at 13h45 ?

Here $N = 8 = 2 * m$. The interpolation polynomial is

$$p(t) = \frac{1}{2}p_{-m}(\exp(-2i\frac{\pi mt}{24}) + \exp(2i\frac{\pi mt}{24})) + \sum_{k=-m+1}^{m-1} p_k \exp(2i\frac{\pi kt}{24})$$

and

$$p_k = \frac{1}{N}\sum_{k=j}^{N-1} T_k \exp(2i\frac{\pi k}{N})$$

Input :

```
q:=1/8*fft([11,10,17,24,32,26,23,19])
```

Output :

```
q:=[20.25,-4.48115530061+1.72227182413*i,-0.375+0.875*i,
-0.768844699385+0.222271824132*i,0.5,
-0.768844699385-0.222271824132*i,
-0.375-0.875*i,-4.48115530061-1.72227182413*i]
```

hence:

- $p_0 = 20.25$
- $p_1 = -4.48115530061 + 1.72227182413 * i = \overline{p_{-1}}$,
- $p_2 = 0.375 + 0.875 * i = \overline{p_{-2}}$,
- $p_3 = -0.768844699385 + 0.222271824132 * i = \overline{p_{-3}}$,
- $p_{-4} = 0.5$

Indeed

$$q = [q_0, ...q_{N-1}] = [p_0, ..p_{\frac{N}{2}-1}, p_{-\frac{N}{2}}, .., p_{-1}] = \frac{1}{N}F_N([y_0, ..y_{N-1}]) = \frac{1}{\mathtt{N}}\mathtt{fft(y)}$$

Input :
`pp:=[q[4],q[5],q[6],q[7],q[0],q[1],q[2],q[3]]`
Here, $p_k = pp[k + 4]$ for $k = -4...3$. It remains to compute the value of the interpolation polynomial at point $t0 = 13.75 = 55/4$.
Input:

```
t0(j):=exp(2*i*pi*(13+3/4)/24*j)
T0:=1/2*pp[0]*(t0(4)+t0(-4))+sum(pp[j+4]*t0(j),j,-3,3)
evalf(re(T0))
```

Output :

```
29.4863181684
```

The temperature is predicted to be equal to 29.49 Celsius degrees.
Input :

```
q1:=[q[4]/2,q[3],q[2],q[1],q[0]/2]
a:=t0(1) (or a:=-exp(i*pi*7/48))
g(x):=r2e(q1,x)
evalf(2*re(g(a)))
```

or :

```
2.0*re(q[0]/2+q[1]*t0(1)+q[2]*t0(2)+q[3]*t0(3)+q[4]/2*t0(4))
```

Output :

```
29.4863181684
```

Remark
Using the Lagrange interpolation polynomial (the polynomial is not periodic), input :

```
l1:=[0,3,6,9,12,15,18,21]
l2:=[11,10,17,24,32,26,23,19]
subst(lagrange(l1,l2,13+3/4),x=13+3/4)
```

Output :

$$\frac{8632428959}{286654464} \simeq 30.1144061688$$

5.23 Signal Processing

5.23.1 Cross-correlation of two signals : `cross_correlation`

`cross_correlation` takes two arguments, a complex vector $\mathbf{v}$ of length n and a complex vector $\mathbf{w}$ of length m. The returned value is the complex vector $\mathbf{z} = \mathbf{v} \star \mathbf{w}$ of length $N = n + m - 1$ which is the cross-correlation of the two input vectors, i.e. such that the following holds :

$$z_k = \sum_{i=k}^{N-1} \overline{v_{i-k}^*}\, w_i^*, \quad k = 0, 1, \ldots, N-1,$$

where

$$\mathbf{v}^* = [v_0, v_1, \ldots, v_{n-1}, \underbrace{0, 0, \ldots, 0}_{m-1}] \quad \text{and} \quad \mathbf{w}^* = [\underbrace{0, 0, \ldots, 0}_{n-1}, w_0, w_1, \ldots, w_{m-1}].$$

For example, input :

```
cross_correlation([1,2],[3,4,5])
```

Output :

```
[6.0,11.0,14.0,5.0]
```

5.23.2 Auto-correlation of a signal : `auto_correlation`

`auto_correlation` takes as argument a complex vector $\mathbf{v}$ of length n and returns its cross-correlation with itself as the vector $\mathbf{v} \star \mathbf{v}$ of length $2\,n - 1$ (see the `cross_correlation` command, section 5.23.1). For example, input :

```
auto_correlation([2,3,4,3,1,4,5,1,3,1])
```

Output :

```
[2.0,9.0,15.0,28.0,37.0,44.0,58.0,58.0,68.0,
91.0,68.0,58.0,58.0,44.0,37.0,28.0,15.0,9.0,2.0]
```

5.23.3 Convolution of two signals : `convolution`

`convolution` takes two arguments, a real vector $\mathbf{v}$ of length n and a real vector $\mathbf{w}$ of length m, and returns their convolution $\mathbf{z} = \mathbf{v} * \mathbf{w}$ which is the vector of length $N = n + m - 1$ defined as :

$$z_k = \sum_{i=0}^{k} v_i\, w_{k-i}, \quad k = 0, 1, \ldots, N-1,$$

such that $v_j = 0$ for $j \geq n$ and $w_j = 0$ for $j \geq m$. For example, input :

```
convolution([1,2,3],[1,-1,1,-1])
```

Output :

```
[1.0,1.0,2.0,-2.0,1.0,-3.0]
```

5.23.4 Low-pass filtering : `lowpass`

`lowpass` takes two or three arguments: a real vector **v** representing the sampled signal, a real number c specifying the cutoff frequency and optionally a samplerate (which defaults to 44100). This implementation is an emulation of a simple first-order lowpass RC filter.

For example, input :

```
f:=unapply(periodic(sign(x),x,-1/880,1/880),x);
s:=2^14*(apply(f,soundsec(1)));
playsnd(lowpass(s,1000))
```

5.23.5 High-pass filtering : `highpass`

`highpass` takes two or three arguments: a real vector **v** representing the sampled signal, a real number c specifying the cutoff frequency and optionally a samplerate (which defaults to 44100). This implementation is an emulation of a simple first-order highpass RC filter.

For example, input :

```
f:=unapply(periodic(sign(x),x,-1/880,1/880),x);
s:=2^14*(apply(f,soundsec(1)));
playsnd(highpass(s,5000))
```

5.23.6 Perform thresholding operations on an array : `threshold`

`threshold` changes the data in an array which does not meet some kind of minimality criterion. It takes the following parameters :

- vector **v** of real or complex numbers
- bound specification `bnd`
- comparison operator (optional)
- `abs[=true,false]` (optional)

Bound specification may be either a single real number b (or an equation `b=value`) or a list of two real numbers l, u (or equations `l=lvalue`, `u=uvalue`). In the latter case a vector **w** is returned, as defined by :

$$w_k = \begin{cases} \texttt{uvalue} \text{ (defaults to } u), & v_k > u, \\ \texttt{lvalue} \text{ (defaults to } l), & v_k < l, \\ v_k, & \text{otherwise} \end{cases}$$

for $k = 0, 1, \ldots, n-1$ where $n =$`size`(**v**) when the element v_k is a real number. If v_k is complex, then $|v_k|$ is compared with u resp. l and the value `uvalue` resp. `lvalue` is multiplied by $\frac{v_k}{|v_k|}$.

In the first case where `bnd` is a number or an equation, the return vector **w** is defined by :

$$w_k = \begin{cases} \texttt{value} \text{ (defaults to } b), & v_k < b, \\ v_k, & \text{otherwise} \end{cases}$$

if $v_k \in \mathbb{R}$ (if v_k is complex, then $|v_k|$ is compared with b and the `value` is multiplied by $\frac{v_k}{|v_k|}$), for $k = 0, 1, \ldots, n-1$. If comparison operator is specified (one of >, <= or >=, must be quoted), it is used instead of < (which is the default) in the above formula. If the fourth argument is specified, the data in $\mathbf{v}$ must be real and the following formula is used for w_k, $k = 0, 1, \ldots, n-1$:

$$w_k = \begin{cases} \texttt{value}, & v_k \geq 0 \text{ and } |v_k| < b, \\ -\texttt{value}, & v_k < 0 \text{ and } |v_k| < b, \\ v_k, & \text{otherwise.} \end{cases}$$

As before, `value` defaults to b and the comparison operator used to test $|v_k|$ against b (by default <) is specified by the third argument.

For example, input :

```
threshold(2,3,1,2,5,4,3,7],3)
```

Output :

```
[3,3,3,3,5,4,3,7]
```

Input :

```
threshold([2,3,1,2,5,4,3,7],3=a,'>=')
```

Output :

```
[2,a,1,2,a,a,a,a]
```

Input :

```
threshold([-2,-3,1,2,5,-4,3,-1],3=0,abs=true)
```

Output :

```
[0,-3,0,0,5,-4,3,0]
```

Input :

```
threshold([-2,-3,1,2,5,-4,3,-1],3=0,'<=',abs=true)
```

Output :

```
[0,0,0,0,5,-4,0,0]
```

Input :

```
threshold([-120,-11,-3,0,7,27,111,234],[-100,100])
```

Output :

```
[-100,-11,-3,0,7,27,100,100]
```

Input :

```
threshold([-120,-11,-3,0,7,27,111,234],[-100=-inf,100=inf])
```

Output :

```
[-infinity,-11,-3,0,7,27,+infinity,+infinity]
```

5.23.7 Bartlett-Hann window function : `bartlett_hann_window`

`bartlett_hann_window` takes as arguments a real vector $\mathbf{v}$ of length n and optionally an interval n_1 `..` n_2 (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = a_0 + a_1 \left| \frac{k}{N-1} - \frac{1}{2} \right| - a_2 \cos\left(\frac{2\,k\,\pi}{N-1} \right)$$

for $k = 0, 1, \ldots, N-1$, where $a_0 = 0.62$, $a_1 = 0.48$ and $a_2 = 0.38$. For example, input :

```
L:=bartlett_hann_window(randvector(1000,0..1)):;
```

followed by `scatterplot(L)`.

5.23.8 Blackman-Harris window function : `blackman_harris_window`

`blackman_harris_window` takes as arguments a real vector $\mathbf{v}$ of length n and optionally an interval n_1 `..` n_2 (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = a_0 - a_1 \cos\left(\frac{2\,k\,\pi}{N-1} \right) + a_2 \cos\left(\frac{4\,k\,\pi}{N-1} \right) - a_3 \cos\left(\frac{6\,k\,\pi}{N-1} \right)$$

for $k = 0, 1, \ldots, N-1$, where $a_0 = 0.35875$, $a_1 = 0.48829$, $a_2 = 0.14128$ and $a_3 = 0.01168$. For example, input :

```
L:=blackman_harris_window(randvector(1000,0..1)):;
```

followed by `scatterplot(L)`.

5.23.9 Blackman window function : `blackman_window`

`blackman_window` takes as arguments a real vector $\mathbf{v}$ of length n and optionally a real number α (by default $\alpha = 0.16$) and/or an interval n_1 `..` n_2 (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \frac{1-\alpha}{2} - \frac{1}{2} \cos\left(\frac{2\,k\,\pi}{N-1} \right) + \frac{\alpha}{2} \cos\left(\frac{4\,k\,\pi}{N-1} \right)$$

for $k = 0, 1, \ldots, N-1$. For example, input :

```
L:=blackman_window(randvector(1000,0..1)):;
```

followed by `scatterplot(L)`.

5.23.10 Bohman window function : bohman_window

bohman_window takes as arguments a real vector $\mathbf{v}$ of length n and optionally an interval $n_1\,..\,n_2$ (with default values $n_1 = 0$ and $n_2 = n-1$), and returns the elementwise product of the vector $[v_{n_1},\ldots,v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = (1-x_k)\,\cos(\pi\,x_k) + \frac{1}{\pi}\,\sin(\pi\,x_k)\,,$$

where $x_k = \left|\frac{2\,k}{N-1} - 1\right|$, for $k = 0,1,\ldots,N-1$. For example, input :

```
L:=bohman_window(randvector(1000,0..1)):;
```

followed by `scatterplot(L)`.

5.23.11 Cosine window function : cosine_window

cosine_window takes as arguments a real vector $\mathbf{v}$ of length n and optionally a positive real number α (by default $\alpha = 1$) and/or an interval $n_1\,..\,n_2$ (with default values $n_1 = 0$ and $n_2 = n-1$), and returns the elementwise product of the vector $[v_{n_1},\ldots,v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \sin^{\alpha}\left(\frac{k\,\pi}{N-1}\right)$$

for $k = 0,1,\ldots,N-1$. For example, input :

```
L:=cosine_window(randvector(1000,0..1),1.5):;
```

followed by `scatterplot(L)`.

5.23.12 Gaussian window function : gaussian_window

gaussian_window takes as arguments a real vector $\mathbf{v}$ of length n and optionally a positive real number $\alpha \leq 0.5$ (by default $\alpha = 0.1$) and/or an interval $n_1\,..\,n_2$ (with default values $n_1 = 0$ and $n_2 = n-1$), and returns the elementwise product of the vector $[v_{n_1},\ldots,v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \exp\left(-\frac{1}{2}\left(\frac{k-(N-1)/2}{\alpha\,(N-1)/2}\right)^2\right)$$

for $k = 0,1,\ldots,N-1$. For example, input :

```
L:=gaussian_window(randvector(1000,0..1),0.4):;
```

followed by `scatterplot(L)`.

5.23.13 Hamming window function : hamming_window

hamming_window takes as arguments a real vector **v** of length n and optionally an interval n_1..n_2 (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector **w** of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \alpha - \beta \cos\left(\frac{2\,k\,\pi}{N-1}\right)$$

for $k = 0, 1, \ldots, N - 1$, where $\alpha = 0.54$ and $\beta = 1 - \alpha = 0.46$. For example, input :

```
L:=hamming_window(randvector(1000,0..1)):;
```

followed by scatterplot(L).

5.23.14 Hann-Poisson window function : hann_poisson_window

hann_poisson_window takes as arguments a real vector **v** of length n and optionally a real number α (by default $\alpha = 1$) and/or an interval n_1..n_2 (with default values $n_1 = 0$ and $n_2 = n-1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector **w** of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \frac{1}{2}\left(1 - \cos\frac{2\,k\,\pi}{N-1}\right) \exp\left(-\frac{\alpha\,|N-1-2\,k|}{N-1}\right)$$

for $k = 0, 1, \ldots, N - 1$. For example, input :

```
L:=hann_poisson_window(randvector(1000,0..1),2):;
```

followed by scatterplot(L).

5.23.15 Hann window function : hann_window

hann_window takes as arguments a real vector **v** of length n and optionally an interval n_1..n_2 (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector **w** of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \sin^2\left(\frac{k\,\pi}{N-1}\right)$$

for $k = 0, 1, \ldots, N - 1$. For example, input :

```
L:=hann_window(randvector(1000,0..1)):;
```

followed by scatterplot(L).

5.23.16 Parzen window function : `parzen_window`

`parzen_window` takes as arguments a real vector $\mathbf{v}$ of length n and optionally an interval $n_1\,..\,n_2$ (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \begin{cases} \left(1 - 6\,x_k^2\,(1 - x_k)\right), & \left|\frac{N-1}{2} - k\right| \leq \frac{N-1}{4}, \\ 2\,(1 - x_k)^3, & \text{otherwise,} \end{cases}$$

where $x_k = \left|1 - \frac{2\,k}{N-1}\right|$, for $k = 0, 1, \ldots, N - 1$. For example, input :

```
L:=parzen_window(randvector(1000,0..1)):;
```

followed by `scatterplot(L)`.

5.23.17 Poisson window function : `poisson_window`

`poisson_window` takes as arguments a real vector $\mathbf{v}$ of length n and optionally a real number α (by default $\alpha = 1$) and/or an interval $n_1\,..\,n_2$ (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \exp\left(-\alpha\left|\frac{2\,k}{N-1} - 1\right|\right)$$

for $k = 0, 1, \ldots, N - 1$. For example, input :

```
L:=poisson_window(randvector(1000,0..1),2):;
```

followed by `scatterplot(L)`.

5.23.18 Riemann window function : `riemann_window`

`riemann_window` takes as arguments a real vector $\mathbf{v}$ of length n and optionally an interval $n_1\,..\,n_2$ (with default values $n_1 = 0$ and $n_2 = n - 1$), and returns the elementwise product of the vector $[v_{n_1}, \ldots, v_{n_2}]$ and the vector $\mathbf{w}$ of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \begin{cases} 1, & k = \frac{N-1}{2}, \\ \frac{\sin(\pi\,x_k)}{\pi\,x_k}, & \text{otherwise,} \end{cases}$$

where $x_k = \frac{2\,k}{N-1} - 1$, for $k = 0, 1, \ldots, N - 1$. For example, input :

```
L:=riemann_window(randvector(1000,0..1)):;
```

followed by `scatterplot(L)`.

5.23.19 Triangular window function : `triangle_window`

`triangle_window` takes as arguments a real vector **v** of length n and optionally an integer $d \in \{-1,0,1\}$ (by default $d = 0$) and/or an interval $n_1 \,..\, n_2$ (with default values $n_1 = 0$ and $n_2 = n-1$), and returns the elementwise product of the vector $[v_{n_1},\dots,v_{n_2}]$ and the vector **w** of length $N = n_2 - n_1 + 1$ defined by

$$w_k = 1 - \left| \frac{n - \frac{N-1}{2}}{\frac{N+d}{2}} \right|$$

for $k = 0,1,\dots,N-1$ (the case $d = -1$ is called the Bartlett window function). For example, input :

```
L:=triangle_window(randvector(1000,0..1),1):;
```

followed by `scatterplot(L)`.

5.23.20 Tukey window function : `tukey_window`

`tukey_window` takes as arguments a real vector **v** of length n and optionally a real number $\alpha \in [0,1]$ (by default $\alpha = 0.5$) and/or an interval $n_1 \,..\, n_2$ (with default values $n_1 = 0$ and $n_2 = n-1$), and returns the elementwise product of the vector $[v_{n_1},\dots,v_{n_2}]$ and the vector **w** of length $N = n_2 - n_1 + 1$ defined by

$$w_k = \begin{cases} \frac{1}{2}\left(1+\cos\left(\pi\left(\frac{k}{\beta}-1\right)\right)\right), & k < \beta, \\ 1, & \beta \le k \le (N-1)\left(1-\frac{\alpha}{2}\right), \\ \frac{1}{2}\left(1+\cos\left(\pi\left(\frac{k}{\beta}-\frac{2}{\alpha}+1\right)\right)\right), & \text{otherwise,} \end{cases}$$

where $\beta = \frac{\alpha\,(N-1)}{2}$, for $k = 0,1,\dots,N-1$. When $\alpha = 0$ the rectangular window function (on-off windowing) is obtained, and the case $\alpha = 1$ corresponds to the Hann window function. For example, input :

```
L:=tukey_window(randvector(1000,0..1),0.4):;
```

followed by `scatterplot(L)`.

5.23.21 Welch window function : `welch_window`

`welch_window` takes as arguments a real vector **v** of length n and optionally an interval $n_1 \,..\, n_2$ (with default values $n_1 = 0$ and $n_2 = n-1$), and returns the elementwise product of the vector $[v_{n_1},\dots,v_{n_2}]$ and the vector **w** of length $N = n_2 - n_1 + 1$ defined by

$$w_k = 1 - \left(\frac{k - \frac{N-1}{2}}{\frac{N-1}{2}} \right)^2$$

for $k = 0,1,\dots,N-1$. For example, input :

```
L:=welch_window(randvector(1000,0..1)):;
```

followed by `scatterplot(L)`.

5.24 Exponentials and Logarithms

5.24.1 Rewrite hyperbolic functions as exponentials : hyp2exp

hyp2exp takes as argument an hyperbolic expression.
hyp2exp rewrites each hyperbolic functions with exponentials (as a rational fraction of one exponential, i.e. WITHOUT linearization).
Input :

```
hyp2exp(sinh(x))
```

Output :

```
(exp(x)-1/(exp(x)))/2
```

5.24.2 Expand exponentials : expexpand

expexpand takes as argument an expression with exponentials.
expexpand expands this expression (rewrites exp of sums as product of exp).
Input :

```
expexpand(exp(3*x)+exp(2*x+2))
```

Output :

```
exp(x)^3+exp(x)^2*exp(2)
```

5.24.3 Expand logarithms : lnexpand

lnexpand takes as argument an expression with logarithms.
lnexpand expands this expression (rewrites ln of products as sum of ln).
Input :

```
lnexpand(ln(3*x^2)+ln(2*x+2))
```

Output :

```
ln(3)+2*ln(x)+ln(2)+ln(x+1)
```

5.24.4 Linearize exponentials : lin

lin takes as argument an expression with exponentials.
lin rewrites hyperbolic functions as exponentials if required, then linearizes this expression (i.e. replace product of exponentials by exponential of sums).
Examples

- Input :

  ```
  lin(sinh(x)^2)
  ```

  Output :

  ```
  1/4*exp(2*x)+1/-2+1/4*exp(-(2*x))
  ```

- Input :

  `lin((exp(x)+1)^3)`

  Output :

  `exp(3*x)+3*exp(2*x)+3*exp(x)+1`

5.24.5 Collect logarithms : `lncollect`

`lncollect` takes as argument an expression with logarithms.
`lncollect` collects the logarithms (rewrites sum of ln as ln of products). It may be a good idea to factor the expression with `factor` before collecting by `lncollect`).
Input :

`lncollect(ln(x+1)+ln(x-1))`

Output :

`log((x+1)*(x-1))`

Input :

`lncollect(exp(ln(x+1)+ln(x-1)))`

Output :

`(x+1)*(x-1)`

Warning!!! For `Xcas`, `log=ln` (use `log10` for 10-base logarithm).

5.24.6 Expand powers : `powexpand`

`powexpand` rewrites a power of a sum as a product of powers.
Input :

`powexpand(a^(x+y))`

Output :

`a^x*a^y`

5.24.7 Rewrite a power as an exponential : `pow2exp`

`pow2exp` rewrites a power as an exponential.
Input :

`pow2exp(a^(x+y))`

Output :

`exp((x+y)*ln(a))`

5.24.8 Rewrite exp(n*ln(x)) as a power : exp2pow

`exp2pow` rewrites expression of the form $\exp(n*\ln(x))$ as a power of x.
Input :

```
exp2pow(exp(n*ln(x)))
```

Output :

```
x^n
```

Note the difference with `lncollect` :
`lncollect(exp(n*ln(x))) = exp(n*log(x))`
`lncollect(exp(2*ln(x))) = exp(2*log(x))`
`exp2pow(exp(2*ln(x))) = x^2`
But :
`lncollect(exp(ln(x)+ln(x))) = x^2`
`exp2pow(exp(ln(x)+ln(x))) = x^(1+1)`

5.24.9 Simplify complex exponentials : tsimplify

`tsimplify` simplifies transcendental expressions by rewriting the expression with complex exponentials.
It is a good idea to try other simplification instructions and call `tsimplify` if they do not work.
Input :

```
tsimplify((sin(7*x)+sin(3*x))/sin(5*x))
```

Output :

```
((exp((i)*x))^4+1)/(exp((i)*x))^2
```

5.25 Polynomials

A polynomial of one variable is represented either by a symbolic expression or by the list of its coefficients in decreasing powers order (dense representation). In the latter case, to avoid confusion with other kinds of list

- use `poly1[...]` as delimiters in inputs
- check for [] [] in `Xcas` output.

Note that polynomials represented as lists of coefficients are always written in decreasing powers order even if `increasing power` is checked in `cas` configuration.

A polynomial of several variables is represented

- by a symbolic expression
- or by a dense recursive 1-d representation like above

- or by a sum of monomials with non-zero coefficients (distributed sparse representation).
  A monomial with several variables is represented by a coefficient and a list of integers (interpreted as powers of a variable list). The delimiters for monomials are `%%%{` and `%%%}`, for example $3x^2y$ is represented by `%%%{3,[2,1]%%%}` with respect to the variable list `[x,y]`).

5.25.1 Convert to a symbolic polynomial : `r2e poly2symb`

`r2e` or `poly2symb` takes as argument

- a list of coefficients of a polynomial (by decreasing order) and a symbolic variable name (by default `x`)
- or a sum of monomials `%%%{coeff,[n1,....nk] %%%}` and a vector of symbolic variables `[x1,...,xk]`.

`r2e` or `poly2symb` transforms the argument into a symbolic polynomial.
Example with univariate polynomials, input :

```
r2e([1,0,-1],x)
```

or :

```
r2e([1,0,-1])
```

or :

```
poly2symb([1,0,-1],x)
```

Output :

```
x*x-1
```

Example with sparse multivariate polynomials, input:

```
poly2symb(%%%{1,[2]%%%}+%%%{-1,[0]%%%},[x])
```

or :

```
r2e(%%%{1,[2]%%%}+%%%{-1,[0]%%%},[x])
```

Output :

```
x^2-1
```

Input :

```
r2e(%%%{1,[2,0]%%%}+%%%{-1,[1,1]%%%}+%%%{2,[0,1]%%%},[x,y])
```

or :

```
poly2symb(%%%{1,[2,0]%%%}+%%%{-1,[1,1]%%%}+%%%{2,[0,1]%%%},[x,y])
```

Output :

```
x^2-x*y+2*y
```

5.25.2 Convert from a symbolic polynomial : `e2r symb2poly`

`e2r` or `symb2poly` takes as argument a symbolic polynomial and either a symbolic variable name (by default `x`) or a list of symbolic variable names.
`e2r` or `symb2poly` transforms the polynomial into a list (dense representation of the univariate polynomial, coefficients written by decreasing order) or into a sum of monomials (sparse representation of multivariate polynomials).
Input :

```
e2r(x^2-1)
```

or :

```
symb2poly(x^2-1)
```

or :

```
symb2poly(y^2-1,y)
```

or :

```
e2r(y^2-1,y)
```

Output :

```
[1,0,-1]
```

Input :

```
e2r(x^2-x*y+y, [x,y])
```

or :

```
symb2poly(x^2-x*y+2*y, [x,y])
```

Output :

```
%%%{1,[2,0]%%%}+%%%{-1,[1,1]%%%}+%%%{2,[0,1]%%%}
```

5.25.3 Coefficients of a polynomial: `coeff coeffs`

`coeff` or `coeffs` takes three arguments : the polynomial, the name of the variable (or the list of the names of variables) and the degree (or the list of the degrees of the variables).
`coeff` or `coeffs` returns the coefficient of the polynomial of the degree given as third argument. If no degree was specified, `coeffs` return the list of the coefficients of the polynomial, including 0 in the univariate dense case and excluding 0 in the multivariate sparse case.
Input :

```
coeff(-x^4+3*x*y^2+x,x,1)
```

Output :

```
3*y^2+1
```

Input :

```
coeff(-x^4+3x*y^2+x,y,2)
```

Output :

```
3*x
```

Input :

```
coeff(-x^4+3x*y^2+x,[x,y],[1,2])
```

Output :

```
3
```

5.25.4 Polynomial degree : degree

`degree` takes as argument a polynomial given by its symbolic representation or by the list of its coefficients.
`degree` returns the degree of this polynomial (highest degree of its non-zero monomials).
Input :

```
degree(x^3+x)
```

Output :

```
3
```

Input :

```
degree([1,0,1,0])
```

Output :

```
3
```

5.25.5 Polynomial valuation : valuation ldegree

`valuation` or `ldegree` takes as argument a polynomial given by a symbolic expression or by the list of its coefficients.
`valuation` or `ldegree` returns the valuation of this polynomial, that is the lowest degree of its non-zero monomials.
Input :

```
valuation(x^3+x)
```

Output :

```
1
```

Input :

```
valuation([1,0,1,0])
```

Output :

```
1
```

5.25.6 Leading coefficient of a polynomial : lcoeff

lcoeff takes as argument a polynomial given by a symbolic expression or by the list of its coefficients.
lcoeff returns the leading coefficient of this polynomial, that is the coefficient of the monomial of highest degree.
Input :

```
lcoeff([2,1,-1,0])
```

Output :

```
2
```

Input :

```
lcoeff(3*x^2+5*x,x)
```

Output :

```
3
```

Input :

```
lcoeff(3*x^2+5*x*y^2,y)
```

Output :

```
5*x
```

5.25.7 Trailing coefficient degree of a polynomial : tcoeff

tcoeff takes as argument a polynomial given by a symbolic expression or by the list of its coefficients.
tcoeff returns the coefficient of the monomial of lowest degree of this polynomial (tcoeff=trailing coefficient).
Input :

```
tcoeff([2,1,-1,0])
```

Output :

```
-1
```

Input :

```
tcoeff(3*x^2+5*x,x)
```

Output :

```
5
```

Input :

```
tcoeff(3*x^2+5*x*y^2,y)
```

Output :

```
3*x^2
```

5.25.8 Evaluation of a polynomial : `peval polyEval`

`peval` or `polyEval` takes as argument a polynomial `p` given by the list of its coefficients and a real `a` .
`peval` or `polyEval` returns the exact or numeric value of `p(a)` using Horner's method.
Input :

```
peval([1,0,-1],sqrt(2))
```

Output :

```
sqrt(2)*sqrt(2)-1
```

Then :

```
normal(sqrt(2)*sqrt(2)-1)
```

Output :

```
1
```

Input :

```
peval([1,0,-1],1.4)
```

Output :

```
0.96
```

5.25.9 Factorize x^n in a polynomial : `factor_xn`

`factor_xn` takes as argument a polynomial `P`.
`factor_xn` returns the polynomial `P` written as the product of its monomial of largest degree x^n (`n=degree(P)`) with a rational fraction having a non-zero finite limit at infinity.
Input :

```
factor_xn(-x^4+3)
```

Output :

```
x^4*(-1+3*x^-4)
```

5.25.10 GCD of the coefficients of a polynomial : `content`

`content` takes as argument a polynomial `P` given by a symbolic expression or by the list of its coefficients.
`content` returns the content of `P`, that is the GCD (greatest common divisor) of the coefficients of `P`.
Input :

```
content(6*x^2-3*x+9)
```

or:

```
content([6,-3,9],x))
```

Output :

```
3
```

5.25.11 Primitive part of a polynomial : `primpart`

`primpart` takes as argument a polynomial `P` given by a symbolic expression or by the list of its coefficients.
`primpart` returns the primitive part of `P`, that is `P` divided by the GCD (greatest common divisor) of its coefficients.
Input :

```
primpart(6x^2-3x+9)
```

or:

```
primpart([6,-3,9],x))
```

Output :

```
2*x^2-x+3
```

5.25.12 Factorization : `collect`

`collect` takes as argument a polynomial or a list of polynomials and optionally an algebraic extension like `sqrt(n)` (for $\sqrt{n}$).
`collect` factorizes the polynomial (or the polynomials in the list) on the field of its coefficient (for example $\mathbb{Q}$) or on the smallest extension containing the optional second argument (e.g. $\mathbb{Q}[\sqrt{n}]$). In complex mode, the field is complexified.
Examples :

- Factorize $x^2 - 4$ over the integers, input :

  ```
  collect(x^2-4)
  ```

  Output in real mode :

  ```
  (x-2)*(x+2)
  ```

- Factorize $x^2 + 4$ over the integers, input :

  ```
  collect(x^2+4)
  ```

  Output in real mode :

  ```
  x^2+4
  ```

  Output in complex mode :

  ```
  (x+2*i)*(x-2*i)
  ```

- Factorize $x^2 - 2$ over the integers, input :

  ```
  collect(x^2-2)
  ```

  Output in real mode :

```
x^2-2
```

But if you input :

```
collect(sqrt(2)*(x^2-2))
```

Output :

```
sqrt(2)*(x-sqrt(2))*(x+sqrt(2))
```

- Factorize over the integers :

$$x^3 - 2x^2 + 1 \text{ and } x^2 - x$$

Input :

```
collect([x^3-2*x^2+1,x^2-x])
```

Output :

```
[(x-1)*(x^2-x-1),x*(x-1)]
```

But, input :

```
collect((x^3-2*x^2+1)*sqrt(5))
```

Output :

```
((19*sqrt(5)-10)*((sqrt(5)+15)*x+7*sqrt(5)-5)*
  ((sqrt(5)+25)*x-13*sqrt(5)-15)*(x-1))/6820
```

Or, input :

```
collect(x^3-2*x^2+1,sqrt(5))
```

Output :

```
((2*sqrt(5)-19)*((sqrt(5)+25)*x-
13*sqrt(5)-15)*(-x+1)*((sqrt(5)+15)*x+7*sqrt(5)-5))/6820
```

5.25.13 Factorization : `factor factoriser`

`factor` takes as argument a polynomial or a list of polynomials and optionally an algebraic extension, e.g. `sqrt(n)`.
`factor` factorizes the polynomial (or the polynomials in the list) on the field of its coefficients (the field is complexified in complex mode) or on the smallest extension containing the optional second argument. Unlike `collect`, `factor` will further factorize each factor of degree 2 if `Sqrt` is checked in the `cas` configuration (see also 5.12.9). You can check the current configuration in the status button under `Xcas` and change the configuration by hitting this status button.
Input :

```
factor(x^2+2*x+1)
```

Output :

```
(x+1)^2
```

Input :

```
factor(x^4-2*x^2+1)
```

Output :

```
(-x+1)^2*(x+1)^2
```

Input :

```
factor(x^3-2*x^2+1)
```

Output if `Sqrt` is not checked in the `cas` configuration :

```
(x-1)*(x^2-x-1)
```

Output if `Sqrt` is checked in the `cas` configuration :

```
(x-1)*(x+(sqrt(5)-1)/2)*(x+(-sqrt(5)-1)/2)
```

Input :

```
factor(x^3-2*x^2+1,sqrt(5))
```

Output :

```
((2*sqrt(5)-19)*((sqrt(5)+15)*x+
7*sqrt(5)-5)*(-x+1)*((sqrt(5)+25)*x-13*sqrt(5)-15))/6820
```

Input :

```
factor(x^2+1)
```

Output in real mode :

```
x^2+1
```

Output in complex mode :

```
((-i)*x+1)*((i)*x+1)
```

5.25.14 Square-free factorization : `sqrfree`

`sqrfree` takes as argument a polynomial.
`sqrfree` factorizes this polynomial as a product of powers of coprime factors, where each factor has roots of multiplicity 1 (in other words, a factor and its derivative are coprime).
Input :

```
sqrfree((x^2-1)*(x-1)*(x+2))
```

Output :

```
(x^2+3*x+2)*(x-1)^2
```

Input :

```
sqrfree((x^2-1)^2*(x-1)*(x+2)^2)
```

Output :

```
(x^2+3*x+2)*(x-1)^3
```

5.25.15 List of factors : `factors`

`factors` has either a polynomial or a list of polynomials as argument.
`factors` returns a list containing the factors of the polynomial and their exponents.
Input :

```
factors(x^2+2*x+1)
```

Output :

```
[x+1,2]
```

Input :

```
factors(x^4-2*x^2+1)
```

Output :

```
[x+1,2,x-1,2]
```

Input :

```
factors([x^3-2*x^2+1,x^2-x])
```

Output :

```
[[x-1,1,x^2-x-1,1],[x,1,x-1,1]]
```

Input :

```
factors([x^2,x^2-1])
```

Output :

```
[[x,2],[x+1,1,x-1,1]]
```

5.25.16 Evaluate a polynomial : `horner`

`horner` takes two arguments : a polynomial `P` given by its symbolic expression or by the list of its coefficients and a number `a`.
`horner` returns `P(a)` computed using Horner's method.
Input :

```
horner(x^2-2*x+1,2)
```

or :

```
horner([1,-2,1],2)
```

Output :

```
1
```

5.25.17 Rewrite in terms of the powers of (x-a) : `ptayl`

`ptayl` is used to rewrite a polynomial `P` depending of `x` in terms of the powers of `(x-a)` (`ptayl` means polynomial Taylor)
`ptayl` takes two arguments: a polynomial `P` given by a symbolic expression or by the list of its coefficients and a number `a`.
`ptayl` returns the polynomial `Q` such that `Q(x-a)=P(x)`
Input :

```
ptayl(x^2+2*x+1,2)
```

Output, the polynomial Q:

```
x^2+6*x+9
```

Input :

```
ptayl([1,2,1],2)
```

Output :

```
[1,6,9]
```

Remark

```
P(x)=Q(x-a)
```

i.e. for the example :
$x^2 + 2x + 1 = (x - 2)^2 + 6(x - 2) + 9$

5.25.18 Compute with the exact root of a polynomial : `rootof`

Let P and Q be two polynomials given by the list of their coefficients then `rootof(P,Q)` gives the value $P(\alpha)$ where α is the root of Q with largest real part (and largest imaginary part in case of equality).
In exact computations, `Xcas` will rewrite rational evaluations of `rootof` as a unique `rootof` with degree(P) <degree(Q). If the resulting rootof is the solution of a second degree equation, it will be simplified.

Example
Let α be the root with largest imaginary part of $Q(x) = x^4 + 10x^2 + 1$ (all roots of Q have real part equal to 0).

- Compute $\frac{1}{\alpha}$. Input :

  ```
  normal(1/rootof([1,0],[1,0,10,0,1]))
  ```

  $P(x) = x$ is represented by [1,0] and α by `rootof([1,0],[1,0,10,0,1])`.
  Output :

  ```
  rootof([[-1,0,-10,0],[1,0,10,0,1]])
  ```

  i.e. :

  $$\frac{1}{\alpha} = -\alpha^3 - 10\alpha$$

- Compute $\alpha 2$. Input :

```
normal(rootof([1,0],[1,0,10,0,1])^2)
```

  or (since $P(x) = x^2$ is represented by [1,0,0]) input

```
normal(rootof([1,0,0],[1,0,10,0,1]))
```

  Output :

```
-5-2*sqrt(6)
```

5.25.19 Exact roots of a polynomial : `roots`

`roots` takes as arguments a symbolic polynomial expression and the name of its variable.
`roots` returns a 2 columns matrix : each row is the list of a root of the polynomial and its multiplicity.
Examples

- Find the roots of $P(x) = x^5 - 2x^4 + x^3$.
  Input :

```
roots(x^5-2*x^4+x^3)
```

  Output :

```
[[8+3*sqrt(7),1],[8-3*sqrt(7),1],[0,3]]
```

- Find the roots of $x^{10} - 15x^8 + 90x^6 - 270x^4 + 405x^2 - 243 = (x^2 - 3)^5$.
  Input :

```
roots(x^10-15*x^8+90*x^6-270*x^4+405*x^2-243)
```

  Output :

```
[[sqrt(3),5],[-(sqrt(3)),5]]
```

- Find the roots of $t^3 - 1$.
  Input :

```
roots(t^3-1,t)
```

  Output :

```
[[(-1+(i)*sqrt(3))/2,1],[(-1-(i)*sqrt(3))/2,1],[1,1]]
```

5.25.20 Coefficients of a polynomial defined by its roots : pcoeff pcoef

pcoeff (or pcoef) takes as argument a list of the roots of a polynomial P.
pcoeff (or pcoef) returns a univariate polynomial having these roots, represented as the list of its coefficients by decreasing order.
Input :

```
pcoef([1,2,0,0,3])
```

Output :

```
[1,-6,11,-6,0,0]
```

i.e. $(x-1)(x-2)(x^2)(x-3) = x^5 - 6x^4 + 11x^3 - 6x^2$.

5.25.21 Truncate of order n : truncate

truncate takes as argument, a polynomial and an integer n.
truncate truncates this polynomial at order n (removing all terms of order greater or equal to n+1).
truncate may be used to transform a series expansion into a polynomial or to compute a series expansion step by step.
Input :

```
truncate((1+x+x^2/2)^3,4)
```

Output :

```
(9*x^4+16*x^3+18*x^2+12*x+4)/4
```

Input :

```
truncate(series(sin(x)),4)
```

Output :

```
(-x^3-(-6)*x)/6
```

Note that the returned polynomial is normalized.

5.25.22 Convert a series expansion into a polynomial : convert convertir

convert, with the option polynom, converts a Taylor series into a polynomial.
It should be used for operations like drawing the graph of the Taylor series of a function near a point.
convert takes two arguments : an expression and the option polynom.
convert replaces the order_size functions by 0 inside the expression.
Input :

```
convert(taylor(sin(x)),polynom)
```

Output :

```
x+1/-6*x^3+1/120*x^5+x^6*0
```

Input :

```
convert(series(sin(x),x=0,6),polynom)
```

Output :

```
x+1/-6*x^3+1/120*x^5+x^7*0
```

5.25.23 Random polynomial : randpoly randPoly

`randpoly` (or `randPoly`) takes two arguments: the name of a variable (by default `x`) and an integer `n` (the order of the arguments is not important).
`randpoly` returns a polynomial with respect to the variable given argument (or `x` if none was provided), of degree the second argument, having as coefficients random integers evenly distributed on -99..+99.
Input :

```
randpoly(t,4)
```

Output for example:

```
-8*t^4-87*t^3-52*t^2+94*t+80
```

Input :

```
randpoly(4)
```

Output for example:

```
70*x^4-46*x^3-7*x^2-24*x+52
```

Input :

```
randpoly(4,u)
```

Output for example:

```
2*u^4+33*u^3-6*u^2-92*u-12
```

5.25.24 Change the order of variables : reorder

`reorder` takes two arguments : an expression and a vector of variable names.
`reorder` expands the expression according to the order of variables given as second argument.
Input :

```
reorder(x^2+2*x*a+a^2+z^2-x*z,[a,x,z])
```

Output :

```
a^2+2*a*x+x^2-x*z+z^2
```

Warning :
The variables must be symbolic (if not, purge them before calling `reorder`)

5.25.25 Random list : `ranm`

`ranm` takes as argument an integer `n`.
`ranm` returns a list of `n` random integers (between -99 and +99). This list can be seen as the coefficients of an univariate polynomial of degree `n-1` (see also 5.42.3 and **??**).
Input :

```
ranm(3)
```

Output :

```
[68,-21,56]
```

5.25.26 Lagrange's polynomial : `lagrange interp`

`lagrange` takes as argument two lists of size `n` (resp. a matrix with two rows and `n` columns) and the name of a variable `var` (by default `x`).
The first list (resp. row) corresponds to the abscissa values x_k ($k = 1..n$), and the second list (resp. row) corresponds to ordinate values y_k ($k = 1..n$).
`lagrange` returns a polynomial expression `P` with respect to `var` of degree `n-1`, such that $P(x_i) = y_i$.
Input :

```
lagrange([[1,3],[0,1]])
```

or :

```
lagrange([1,3],[0,1])
```

Output :

```
(x-1)/2
```

since $\frac{x-1}{2} = 0$ for $x = 1$, and $\frac{x-1}{2} = 1$ for $x = 3$.
Input :

```
lagrange([1,3],[0,1],y)
```

Output :

```
(y-1)/2
```

Warning
`f:=lagrange([1,2],[3,4],y)` does not return a function but an expression with respect to y. To define f as a function, input

```
f:=unapply(lagrange([1,2],[3,4],x),x)
```

Avoid `f(x):=lagrange([1,2],[3,4],x)` since the Lagrange polynomial would be computed each time `f` is called (indeed in a function definition, the second member of the assignment is not evaluated). Note also that
`g(x):=lagrange([1,2],[3,4])` would not work since the default argument of `lagrange` would be global, hence not the same as the local variable used for the definition of `g`.

5.25.27 Trigonometric interpolation : `triginterp`

`triginterp(y,x=a..b)` or `triginterp(y,a,b,x)` returns the trigonometric polynomial that interpolates data given in the list **y**. It is assumed that the list **y** contains ordinate components of the points with equidistant abscissa components between a and b such that the first element from **y** corresponds to a and the last element to b.

For example, **y** may be a list of experimental measurements of some quantity taken in regular intervals, with the first observation in the moment $t = a$ and the last observation in the moment $t = b$. The resulting trigonometric polynomial has the period

$$T = \frac{n\,(b-a)}{n-1},$$

where n is the number of observations (n=`size(y)`). For example, assume that the following data is obtained by measuring the temperature every three hours:

hour of the day	0	3	6	9	12	15	18	21
temperature (deg C)	11	10	17	24	32	26	23	19

Furthermore, assume that an estimate of the temperature at 13:45 is required. To obtain a trigonometric interpolation of the data, input :

```
tp:=triginterp([11,10,17,24,32,26,23,19],x=0..21)
```

Output :

```
81/4+(-21*sqrt(2)-42)/8*cos(pi/12*x)+
(-11*sqrt(2)-12)/8*sin(pi/12*x)+3/4*cos(pi/6*x)
-7/4*sin(pi/6*x)+(21*sqrt(2)-42)/8*cos(pi/4*x)
+(-11*sqrt(2)+12)/8*sin(pi/4*x)+1/2*cos(pi/3*x)
```

Now a temperature at 13:45 hrs can be approximated with the value of `tp` for $x = 13.75$. Input :

```
tp | x=13.75
```

Output :

```
29.4863181684
```

If one of the input parameters is inexact, the result will be inexact too. For example, input :

```
Digits:=3;
triginterp([11,10,17,24,32,26,23,19],x=0..21.0)
```

Output :

```
0.5*cos(1.05*x)-1.54*cos(0.785*x)+0.75*cos(0.524*x)
-8.96*cos(0.262*x)-0.445*sin(0.785*x)-1.75*sin(0.524*x)
-3.44*sin(0.262*x)+20.2
```

5.25.28 Natural splines: `spline`

Definition

Let σ_n be a subdivision of a real interval $[a, b]$:

$$a = x_0, \quad x_1, \quad ..., \quad x_n = b$$

s is a spline function of degree l, if s is a function from $[a, b]$ to $\mathbb{R}$ such that :

- s has continuous derivatives up to the order $l - 1$,
- on each interval of the subdivision, s is a polynomial of degree less or equal than l.

Theorem

The set of spline functions of degree l on σ_n is an $\mathbb{R}$-vector subspace of dimension $n + l$.

Proof

On $[a, x_1]$, s is a polynomial A of degree less or equal to l, hence on $[a, x_1]$, $s = A(x) = a_0 + a_1x + ...a_lx^l$ and A is a linear combination of $1, x, ...x^l$.
On $[x_1, x_2]$, s is a polynomial B of degree less or equal to l, hence on $[x_1, x_2]$, $s = B(x) = b_0 + b_1x + ...b_lx^l$.
s has continuous derivatives up to order $l - 1$, hence :

$$\forall 0 \leq j \leq l - 1, \quad B^{(j)}(x_1) - A^{(j)}(x_1) = 0$$

therefore $B(x) - A(x) = \alpha_1(x - x_1)^l$ or $B(x) = A(x) + \alpha_1(x - x_1)^l$.
Define the function :

$$\mathrm{q}_1(x) = \begin{cases} 0 & \text{on} \quad [a, x_1] \\ (x - x_1)^l & \text{on} \quad [x_1, b] \end{cases}$$

Hence :

$$s|_{[a,x_2]} = a_0 + a_1x + ...a_lx^l + \alpha_1 q_1(x)$$

On $[x_2, x_3]$, s is a polynomial C of degree less or equal than l, hence on $[x_2, x_3]$, $s = C(x) = c_0 + c_1x + ...c_lx^l$.
s has continuous derivatives until $l - 1$, hence :

$$\forall 0 \leq j \leq l - 1, \quad C^{(j)}(x_2) - B^{(j)}(x_2) = 0$$

therefore $C(x) - B(x) = \alpha_2(x - x_2)^l$ or $C(x) = B(x) + \alpha_2(x - x_2)^l$.
Define the function :

$$\mathrm{q}_2(x) = \begin{cases} 0 & \text{on} \quad [a, x_2] \\ (x - x_2)^l & \text{on} \quad [x_2, b] \end{cases}$$

Hence : $s|_{[a,x_3]} = a_0 + a_1x + ...a_lx^l + \alpha_1 q_1(x) + \alpha_2 q_2(x)$
And so on, the functions are defined by :

$$\forall 1 \leq j \leq n - 1, \mathrm{q}_j(x) = \begin{cases} 0 & \text{on} \quad [a, x_j] \\ (x - x_j)^l & \text{on} \quad [x_j, b] \end{cases}$$

hence,

$$s|_{[a,b]} = a_0 + a_1x + ...a_lx^l + \alpha_1 q_1(x) + ... + \alpha_{n-1}q_{n-1}(x)$$

and s is a linear combination of $n + l$ independent functions $1, x, ..x^l, q_1, ..q_{n-1}$.

Interpolation with spline functions

If we want to interpolate a function f on σ_n by a spline function s of degree l, then s must verify $s(x_k) = y_k = f(x_k)$ for all $0 \leq k \leq n$. Hence there are $n+1$ conditions, and $l-1$ degrees of liberty. We can therefore add $l-1$ conditions, these conditions are on the derivatives of s at a and b.

Hermite interpolation, natural interpolation and periodic interpolation are three kinds of interpolation obtained by specifying three kinds of constraints. The unicity of the solution of the interpolation problem can be proved for each kind of constraints.

If l is odd ($l = 2m-1$), there are $2m-2$ degrees of freedom. The constraints are defined by :

- Hermite interpolation

$$\forall 1 \leq j \leq m-1, \quad s^{(j)}(a) = f^{(j)}(a), s^{(j)}(b) = f^{(j)}(b)$$

- Natural interpolation

$$\forall m \leq j \leq 2m-2, \quad s^{(j)}(a) = s^{(j)}(b) = 0$$

- periodic interpolation

$$\forall 1 \leq j \leq 2m-2, \quad s^{(j)}(a) = s^{(j)}(b)$$

If l is even ($l = 2m$), there are $2m-1$ degrees of liberty. The constraints are defined by :

- Hermite interpolation

$$\forall 1 \leq j \leq m-1, \quad s^{(j)}(a) = f^{(j)}(a), s^{(j)}(b) = f^{(j)}(b)$$

  and

$$s^{(m)}(a) = f^{(m)}(a)$$

- Natural interpolation

$$\forall m \leq j \leq 2m-2, \quad s^{(j)}(a) = s^{(j)}(b) = 0$$

  and

$$s^{(2m-1)}(a) = 0$$

- Periodic interpolation

$$\forall 1 \leq j \leq 2m-1, \quad s^{(j)}(a) = s^{(j)}(b)$$

A natural spline is a spline function which verifies the natural interpolation constraints.

`spline` takes as arguments a list of abscissa (by increasing order), a list of ordinates, a variable name, and a degree.
`spline` returns the natural spline function (with the specified degree and crossing points) as a list of polynomials, each polynomial being valid on an interval.

Examples:

1. a natural spline of degree 3, crossing through the points $x_0 = 0, y_0 = 1$, $x_1 = 1, y_1 = 3$ and $x_2 = 2, y_2 = 0$, input :

```
spline([0,1,2],[1,3,0],x,3)
```

   Output is a list of two polynomial expressions of x :

$$[-5{*}x^3/4{+}13{*}x/4{+}1, \quad 5{*}(x{-}1)^3/4{-}15{*}(x{-}1)^2/4{+}(x{-}1)/{-}2{+}3]$$

   defined respectively on the intervals $[0, 1]$ and $[1, 2]$.

2. a natural spline of degree 4, crossing through the points $x_0 = 0, y_0 = 1$, $x_1 = 1, y_1 = 3$, $x_2 = 2, y_2 = 0$ and $x_3 = 3, y_3 = -1$, input :

```
spline([0,1,2,3],[1,3,0,-1],x,4)
```

   Output is a list of three polynomial functions of x :

$$[(-62 * x^4 + 304 * x)/121 + 1,$$

$$(201 * (x-1)^4 - 248 * (x-1)^3 - 372 * (x-1)^2 + 56 * (x-1))/121 + 3,$$

$$(-139 * (x-2)^4 + 556 * (x-2)^3 + 90 * (x-2)^2 + -628 * (x-2))/121]$$

   defined respectively on the intervals $[0, 1]$, $[1, 2]$ and $[2, 3]$.

3. The natural spline interpolation of cos on $[0, \pi/2, 3\pi/2]$, input :

```
spline([0,pi/2,3*pi/2],cos([0,pi/2,3*pi/2]),x,3)
```

   Output :

$$[((3 * \pi^3 + (-7 * \pi^2) * x + 4 * x^3) * 1/3)/(\pi^3),$$

$$((15 * \pi^3 + (-46 * \pi^2) * x + 36 * \pi * x^2 - 8 * x^3) * 1/12)/(\pi^3)]$$

5.25.29 Rational interpolation : `thiele`

`thiele` takes as the first argument a matrix `data` of type $n \times 2$ where that i-th row holds coordinates x and y of i-th point, respectively. The second argument is `v`, which may be an identifier, number or any symbolic expression. Function returns $R(v)$ where R is the rational interpolant. Instead of a single matrix `data`, two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ may be given (in this case, `v` is given as the third argument).

This method computes Thiele interpolated continued fraction based on the concept of reciprocal differences.

It is not guaranteed that R is continuous, i.e. it may have singularities in the shortest segment which contains all components of $\mathbf{x}$.

Examples

Input :

```
thiele([[1,3],[2,4],[4,5],[5,8]],x)
```

Output :

```
(19*x^2-45*x-154)/(18*x-78)
```

Input :

```
thiele([1,2,a],[3,4,5],3)
```

Output :

```
(13*a-29)/(3*a-7)
```

In the following example, data is obtained by sampling the function $f(x) = (1-x^4)\,\mathrm{e}^{1-x^3}$.
Input :

```
data_x:=[-1,-0.75,-0.5,-0.25,0,
0.25,0.5,0.75,1,1.25,1.5,1.75,2];
data_y:=[0.0,2.83341735599,2.88770329586,
2.75030303645,2.71828182846,2.66568510781,
2.24894558809,1.21863761951,0.0,-0.555711613283,
-0.377871362418,-0.107135851128,-0.0136782294833];
thiele(data_x,data_y,x)
```

Output :

```
(-1.55286115659*x^6+5.87298387514*x^5-5.4439152812*x^4
+1.68655817708*x^3-2.40784868317*x^2-7.55954205222*x
+9.40462512097)/(x^6-1.24295718965*x^5-1.33526268624*x^4
+4.03629272425*x^3-0.885419321*x^2-2.77913222418*x
+3.45976823393)
```

5.26 Arithmetic and polynomials

Polynomials are represented by expressions or by list of coefficients by decreasing power order. In the first case, for instructions requiring a main variable (like extended gcd computations), the variable used by default is x if not specified. For modular coefficients in $\mathbb{Z}/n\mathbb{Z}$, use `% n` for each coefficient of the list or apply it to the expression defining the polynomial.

5.26.1 The divisors of a polynomial : `divis`

`divis` takes as argument a polynomial (or a list of polynomials) and returns the list of the divisors of the polynomial(s).
Input :

```
divis(x^4-1)
```

Output :

```
[1,x^2+1,x+1,(x^2+1)*(x+1),x-1,(x^2+1)*(x-1),
(x+1)*(x-1),(x^2+1)*(x+1)*(x-1)]
```

Input :

```
divis([x^2,x^2-1])
```

Output :

```
[[1,x,x^2],[1,x+1,x-1,(x+1)*(x-1)]]
```

5.26.2 Euclidean quotient : quo

quo returns the euclidean quotient q of the Euclidean division between two polynomials (decreasing power order). If the polynomials are represented as expressions, the variable may be specified as a third argument.
Input :

```
quo(x^2+2*x +1,x)
```

Output :

```
x+2
```

Input :

```
quo(y^2+2*y +1,y,y)
```

Output :

```
y+2
```

In list representation, the quotient of $x^2 + 2x + 4$ by $x^2 + x + 2$ one can also input :

```
quo([1,2,4],[1,1,2])
```

Output :

```
[1]
```

that is to say the polynomial 1.

5.26.3 Euclidean quotient : Quo

Quo is the inert form of quo.
Quo returns the euclidean quotient between two polynomials (decreasing power division) without evaluation. It is used when Xcas is in Maple mode to compute the euclidean quotient of the division of two polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$ using Maple-like syntax.
In Xcas mode, input :

```
Quo(x^2+2*x+1,x)
```

Output :

```
quo(x^2+2*x+1,x)
```

In `Maple` mode, input :

```
Quo(x^3+3*x,2*x^2+6*x+5) mod 5
```

Output :

```
-(2)*x+1
```

The division was done using modular arithmetic, unlike with

```
quo(x^3+3*x,2*x^2+6*x+5) mod 5
```

where the division is done in $\mathbb{Z}[X]$ and reduced after to:

```
3*x-9
```

If `Xcas` is not in Maple mode, polynomial division in $\mathbb{Z}/p\mathbb{Z}[X]$ is done e.g. by :

```
quo((x^3+3*x)% 5,(2x^2+6x+5)%5)
```

5.26.4 Euclidean remainder : `rem`

`rem` returns the euclidean remainder between two polynomials (decreasing power division). If the polynomials are represented as expressions, the variable may be specified as a third argument.
Input :

```
rem(x^3-1,x^2-1)
```

Output :

```
x-1
```

To have the remainder of $x^2 + 2x + 4$ by $x^2 + x + 2$ we can also input :

```
rem([1,2,4],[1,1,2])
```

Output :

```
[1,2]
```

i.e. the polynomial $x + 2$.

5.26.5 Euclidean remainder: Rem

Rem is the inert form of rem.
Rem returns the euclidean remainder between two polynomials (decreasing power division) without evaluation. It is used when Xcas is in Maple mode to compute the euclidean remainder of the division of two polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$ using Maple-like syntax.
In Xcas mode, input :

```
Rem(x^3-1,x^2-1)
```

Output :

```
rem(x^3-1,x^2-1)
```

In Maple mode, input :

```
Rem(x^3+3*x,2*x^2+6*x+5) mod 5
```

Output :

```
2*x
```

The division was done using modular arithmetic, unlike with

```
rem(x^3+3*x,2*x^2+6*x+5) mod 5
```

where the division is done in $\mathbb{Z}[X]$ and reduced after to:

```
12*x
```

If Xcas is not in Maple mode, polynomial division in $\mathbb{Z}/p\mathbb{Z}[X]$ is done e.g. by :

```
rem((x^3+3*x)% 5,(2x^2+6x+5)%5)
```

5.26.6 Quotient and remainder : quorem divide

quorem (or divide) returns the list of the quotient and the remainder of the euclidean division (by decreasing power) of two polynomials.
Input :

```
quorem([1,2,4],[1,1,2])
```

Output :

```
[poly1[1],poly1[1,2]]
```

Input :

```
quorem(x^3-1,x^2-1)
```

Output :

```
[x,x-1]
```

5.26.7 GCD of two polynomials with the Euclidean algorithm: gcd

gcd denotes the gcd (greatest common divisor) of two polynomials (or of a list of polynomials or of a sequence of polynomials) (see also 5.6.2 for GCD of integers).

Examples

Input :

```
gcd(x^2+2*x+1,x^2-1)
```

Output :

```
x+1
```

Input :

```
gcd(x^2-2*x+1,x^3-1,x^2-1,x^2+x-2)
```

or

```
gcd([x^2-2*x+1,x^3-1,x^2-1,x^2+x-2])
```

Output :

```
x-1
```

For polynomials with modular coefficients, input e.g. :

```
gcd((x^2+2*x+1) mod 5,(x^2-1) mod 5)
```

Output :

```
x % 5
```

Note that :

```
gcd(x^2+2*x+1,x^2-1) mod 5
```

will output :

```
1
```

since the mod operation is done after the GCD is computed in $\mathbb{Z}[X]$.

5.26.8 GCD of two polynomials with the Euclidean algorithm : Gcd

Gcd is the inert form of gcd. Gcd returns the gcd (greatest common divisor) of two polynomials (or of a list of polynomials or of a sequence of polynomials) without evaluation. It is used when Xcas is in Maple mode to compute the gcd of polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$ using Maple-like syntax.

Input in Xcas mode :

```
Gcd(x^3-1,x^2-1)
```

Output :

```
gcd(x^3-1,x^2-1)
```

Input in Maple mode :

```
Gcd(x^2+2*x,x^2+6*x+5) mod 5
```

Output :

```
1
```

5.26.9 Choosing the GCD algorithm of two polynomials : `ezgcd heugcd modgcd psrgcd`

`ezgcd heugcd modgcd psrgcd` denote the gcd (greatest common divisor) of two univariate or multivariate polynomials with coefficients in $\mathbb{Z}$ or $\mathbb{Z}[i]$ using a specific algorithm :

- `ezgcd` ezgcd algorithm,
- `heugcd` heuristic gcd algorithm,
- `modgcd` modular algorithm,
- `psrgcd` sub-resultant algorithm.

Input :

```
ezgcd(x^2-2*x*y+y^2-1,x-y)
```

or :

```
heugcd(x^2-2*x*y+y^2-1,x-y)
```

or :

```
modgcd(x^2-2*x*y+y^2-1,x-y)
```

or :

```
psrgcd(x^2-2*x*y+y^2-1,x-y)
```

Output :

```
1
```

Input :

```
ezgcd((x+y-1)*(x+y+1),(x+y+1)^2)
```

or :

```
heugcd((x+y-1)*(x+y+1),(x+y+1)^2)
```

or :

```
modgcd((x+y-1)*(x+y+1),(x+y+1)^2)
```

Output :

```
x+y+1
```

Input :

```
psrgcd((x+y-1)*(x+y+1),(x+y+1)^2)
```

Output :

```
-x-y-1
```

Input :

```
ezgcd((x+1)^4-y^4,(x+1-y)^2)
```

Output :

```
"GCD not successful Error:  Bad Argument Value"
```

But input :

```
heugcd((x+1)^4-y^4,(x+1-y)^2)
```

or :

```
modgcd((x+1)^4-y^4,(x+1-y)^2)
```

or :

```
psrgcd((x+1)^4-y^4,(x+1-y)^2)
```

Output :

```
x-y+1
```

5.26.10 LCM of two polynomials : `lcm`

`lcm` returns the LCM (Least Common Multiple) of two polynomials (or of a list of polynomials or of a sequence of polynomials) (see 5.6.5 for LCM of integers).
Input :

```
lcm(x^2+2*x+1,x^2-1)
```

Output :

```
(x+1)*(x^2-1)
```

Input :

```
lcm(x,x^2+2*x+1,x^2-1)
```

or

```
lcm([x,x^2+2*x+1,x^2-1])
```

Output :

```
(x^2+x)*(x^2-1)
```

5.26.11 Bézout's Identity : `egcd gcdex`

This function computes the polynomial coefficients of Bézout's Identity (also known as Extended Greatest Common Divisor). Given two polynomials $A(x), B(x)$, `egcd` computes 3 polynomials $U(x), V(x)$ and $D(x)$ such that :

$$U(x) * A(x) + V(x) * B(x) = D(x) = GCD(A(x), B(x))$$

`egcd` takes 2 or 3 arguments: the polynomials A and B as expressions in terms of a variable, if the variable is not specified it will default to x. Alternatively, A and B may be given as list-polynomials.
Input :

```
egcd(x^2+2*x+1,x^2-1)
```

Output :

```
[1,-1,2*x+2]
```

Input :

```
egcd([1,2,1],[1,0,-1])
```

Output :

```
[[1],[-1],[2,2]]
```

Input :

```
egcd(y^2-2*y+1,y^2-y+2,y)
```

Output :

```
[y-2,-y+3,4]
```

Input :

```
egcd([1,-2,1],[1,-1,2])
```

Output :

```
[[1,-2],[-1,3],[4]]
```

5.26.12 Solving au+bv=c over polynomials: `abcuv`

`abcuv` solves the polynomial equation

$$C(x) = U(x) * A(x) + V(x) * B(x)$$

where A, B, C are given polynomials and U and V are unknown polynomials. C must be a multiple of the gcd of A and B for a solution to exist. `abcuv` takes 3 expressions as argument, and an optional variable specification (which defaults to x) and returns a list of 2 expressions (U and V). Alternatively, the polynomials A, B, C may be entered as list-polynomials.

Input :

```
abcuv(x^2+2*x+1 ,x^2-1,x+1)
```

Output :

```
[1/2,1/-2]
```

Input :

```
abcuv(x^2+2*x+1 ,x^2-1,x^3+1)
```

Output :

```
[1/2*x^2+1/-2*x+1/2,-1/2*x^2-1/-2*x-1/2]
```

Input :

```
abcuv([1,2,1],[1,0,-1],[1,0,0,1])
```

Output :

```
[poly1[1/2,1/-2,1/2],poly1[1/-2,1/2,1/-2]]
```

5.26.13 Chinese remainders : chinrem

chinrem takes two lists as argument, each list being made of 2 polynomials (either expressions or as a list of coefficients in decreasing order). If the polynomials are expressions, an optional third argument may be provided to specify the main variable, by default x is used. chinrem([A,R],[B,Q]) returns the list of two polynomials P and S such that :

$$S = RQ, \quad P = A \pmod{R}, \quad P = B \pmod{Q}$$

If R and Q are coprime, a solution P always exists and all the solutions are congruent modulo S=R*Q. For example, assume we want to solve :

$$\left\{ \begin{array}{lll} P(x) = & x & \bmod (x^2+1) \\ P(x) = & x-1 & \bmod (x^2-1) \end{array} \right.$$

Input :

```
chinrem([[1,0],[1,0,1]],[[1,-1],[1,0,-1]])
```

Output :

```
[[1/-2,1,1/-2],[1,0,0,0,-1]]
```

or :

```
chinrem([x,x^2+1],[x-1,x^2-1])
```

Output :

```
[1/-2*x^2+x+1/-2,x^4-1]
```

hence $P(x) = -\frac{x^2-2.x+1}{2} \pmod{x^4-1}$
Another example, input :

```
chinrem([[1,2],[1,0,1]],[[1,1],[1,1,1]])
```

Output :

```
[[-1,-1,0,1],[1,1,2,1,1]]
```

or :

```
chinrem([y+2,y^2+1],[y+1,y^2+y+1],y)
```

Output :

```
[-y^3-y^2+1,y^4+y^3+2*y^2+y+1]
```

5.26.14 Cyclotomic polynomial : `cyclotomic`

`cyclotomic` takes an integer n as argument and returns the list of the coefficients of the cyclotomic polynomial of index n. This is the polynomial having the n-th primitive roots of unity as zeros (an n-th root of unity is primitive if the set of its powers is the set of all the n-th roots of unity).

For example, let $n = 4$, the fourth roots of unity are: $\{1, i, -1, -i\}$ and the primitive roots are: $\{i, -i\}$. Hence, the cyclotomic polynomial of index 4 is $(x - i).(x + i) = x^2 + 1$. Verification:

```
cyclotomic(4)
```

Output :

```
[1,0,1]
```

Another example, input :

```
cyclotomic(5)
```

Output :

```
[1,1,1,1,1]
```

Hence, the cyclotomic polynomial of index 5 is $x^4+x^3+x^2+x+1$ which divides $x^5 - 1$ since $(x - 1) * (x^4 + x^3 + x^2 + x + 1) = x^5 - 1$.
Input :

```
cyclotomic(10)
```

Output :

```
[1,-1,1,-1,1]
```

Hence, the cyclotomic polynomial of index 10 is $x^4 - x^3 + x^2 - x + 1$ and

$$(x^5 - 1) * (x + 1) * (x^4 - x^3 + x^2 - x + 1) = x^{10} - 1$$

Input :

```
cyclotomic(20)
```

Output :

```
[1,0,-1,0,1,0,-1,0,1]
```

Hence, the cyclotomic polynomial of index 20 is $x^8 - x^6 + x^4 - x^2 + 1$ and

$$(x^{10} - 1) * (x^2 + 1) * (x^8 - x^6 + x^4 - x^2 + 1) = x^{20} - 1$$

5.26.15 Sturm sequences and number of sign changes of P on $(a,\ b]$: `sturm`

`sturm` takes two or four arguments : P a polynomial expression or P/Q a rational fraction and a variable name or P a polynomial expression, a variable name and two real or complex numbers a and b.

If `sturm` takes two arguments, `sturm` returns the list of the Sturm sequences and multiplicities of the square-free factors of P (or P/Q) (in this case `sturm` behaves like `sturmseq`).

If `sturm` takes four arguments, it behaves like `sturmab` :

- if a and b are reals, `sturm` returns the number of sign changes of P on $(a,\ b]$
- if a or b are complex, `sturm` returns the number of complex roots of P in the rectangle having a and b as opposite vertices.

Input :

```
sturm(2*x^3+2,x)
```

Output :

```
[2,[[1,0,0,1],[3,0,0],-9],1]
```

Input :

```
sturm((2*x^3+2)/(x+2),x)
```

Output :

```
[2,[[1,0,0,1],[3,0,0],-9],1,[[1,2],1]]
```

Input :

```
sturm(x^2*(x^3+2),x,-2,0)
```

Output :

```
1
```

5.26.16 Number of zeros in $[a, b)$: `sturmab`

`sturmab` takes four arguments: a polynomial expression P, a variable name and two real or complex numbers a and b

- if a and b are reals, `sturmab` returns the number of sign changes of P on $(a,\ b]$. In other words, it returns the number of zeros in $[a, b)$ of the polynomial P/G where $G = \text{gcd}(P, \text{diff}(P))$.
- if a or b are complex, `sturmab` returns the number of complex roots of P in the rectangle having a and b as opposite vertices.

Input :

```
sturmab(x^2*(x^3+2),x,-2,0)
```

Output :

```
1
```

Input :

```
sturmab(x^3-1,x,-2-i,5+3i)
```

Output :

```
3
```

Input :

```
sturmab(x^3-1,x,-i,5+3i)
```

Output :

```
1
```

Warning !!!!
P is defined by its symbolic expression.
Input :
`sturmab([1,0,0,2,0,0],x,-2,0)`,
Output :
`Bad argument type`.

5.26.17 Sturm sequences : `sturmseq`

`sturmseq` takes as argument, a polynomial expression P or a rational fraction P/Q and returns the list of the Sturm sequences of the square-free factors of odd multiplicity of P (or of P/Q). For F a square-free factor of odd multiplicity, the Sturm sequence $R_1, R_2, ...$ is made from F, F' by a recurrence relation :

- R_1 is the opposite of the euclidean division remainder of F by F' then,
- R_2 is the opposite of the euclidean division remainder of F' by R_1,
- ...
- and so on until $R_k = 0$.

Input :

```
sturmseq(2*x^3+2)
```

or

```
sturmseq(2*y^3+2,y)
```

Output :

```
[2,[[1,0,0,1],[3,0,0],-9],1]
```

The first term gives the content of the numerator (here 2), then the Sturm sequence (in list representation) $[x^3 + 1, 3x^2, -9]$.
Input :

```
sturmseq((2*x^3+2)/(3*x^2+2),x)
```

Output :

```
[2,[[1,0,0,1],[3,0,0],-9],1,[1,[[3,0,2],[6,0],-72]]
```

The first term gives the content of the numerator (here 2), then the Sturm sequence of the numerator ([[1,0,0,1],[3,0,0],-9]), then the content of the denominator (here 1) and the Sturm sequence of the denominator ([[3,0,2],[6,0],-72]). As expressions, $[x^3+1, 3x^2, -9]$ is the Sturm sequence of the numerator and $[3x^2+2, 6x, -72]$ is the Sturm sequence of the denominator.
Input :

```
sturmseq((x^3+1)^2,x)
```

Output :

```
[1,1]
```

Indeed $F = 1$.
Input :

```
sturmseq(3*(3*x^3+1)/(2*x+2),x)
```

Output :

```
[3,[[3,0,0,1],[9,0,0],-81],2,[[1,1],1]]
```

The first term gives the content of the numerator (here `3`),
the second term gives the Sturm sequence of the numerator (here `3x^3+1, 9x^2, -81`),
the third term gives the content of the denominator (here `2`),
the fourth term gives the Sturm sequence of the denominator (`x+1,1`).
Warning !!!!
P is defined by its symbolic expression.
Input :

```
sturmseq([1,0,0,1],x)
```

Output :

```
Bad argument type.
```

5.26.18 Sylvester matrix of two polynomials : `sylvester`

`sylvester` takes two polynomials as arguments.
`sylvester` returns the Sylvester matrix S of these polynomials.
If $A(x) = \sum_{i=0}^{i=n} a_i x^i$ and $B(x) = \sum_{i=0}^{i=m} b_i x^i$ are 2 polynomials, their Sylvester matrix S is a square matrix of size `m+n` where `m=degree(B(x))` and `n=degree(A(x))`.
The `m` first lines are made with the $A(x)$ coefficients, so that :

$$\begin{pmatrix} s_{11}=a_n & s_{12}=a_{n-1} & \cdots & s_{1(n+1)}=a_0 & 0 & \cdots & 0 \\ s_{21}=0 & s_{22}=a_n & \cdots & s_{2(n+1)}=a_1 & s_{2(n+2)}=a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{m1}=0 & s_{m2}=0 & \cdots & s_{m(n+1)}=a_{m-1} & s_{m(n+2)}=a_{m-2} & \cdots & a_0 \end{pmatrix}$$

and the n further lines are made with the $B(x)$ coefficients, so that :

$$\begin{pmatrix} s_{(m+1)1} = b_m & s_{(m+1)2} = b_{m-1} & \cdots & s_{(m+1)(m+1)} = b_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{(m+n)1} = 0 & s_{(m+n)2} = 0 & \cdots & s_{(m+n)(m+1)} = b_{n-1} & b_{n-2} & \cdots & b_0 \end{pmatrix}$$

Input :

```
sylvester(x^3-p*x+q,3*x^2-p,x)
```

Output :

```
[[1,0,-p,q,0],[0,1,0,-p,q],[3,0,-p,0,0],
     [0,3,0,-p,0],[0,0,3,0,-p]]
```

Input :

```
det([[1,0,-p,q,0],[0,1,0,-p,q],[3,0,-p,0,0],
     [0,3,0,-p,0],[0,0,3,0,-p]])
```

Output :

```
-4*p^3-27*q^2
```

5.26.19 Resultant of two polynomials : `resultant`

`resultant` takes as argument two polynomials and returns the resultant of the two polynomials.
The resultant of two polynomials is the determinant of their Sylvester matrix S. The Sylvester matrix S of two polynomials $A(x) = \sum_{i=0}^{i=n} a_i x^i$ and $B(x) = \sum_{i=0}^{i=m} b_i x^i$ is a square matrix with $m+n$ rows and columns; its first m rows are made from the coefficients of $A(X)$:

$$\begin{pmatrix} s_{11} = a_n & s_{12} = a_{n-1} & \cdots & s_{1(n+1)} = a_0 & 0 & \cdots & 0 \\ s_{21} = 0 & s_{22} = a_n & \cdots & s_{2(n+1)} = a_1 & s_{2(n+2)} = a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{m1} = 0 & s_{m2} = 0 & \cdots & s_{m(n+1)} = a_{m-1} & s_{m(n+2)} = a_{m-2} & \cdots & a_0 \end{pmatrix}$$

and the following n rows are made in the same way from the coefficients of $B(x)$:

$$\begin{pmatrix} s_{(m+1)1} = b_m & s_{(m+1)2} = b_{m-1} & \cdots & s_{(m+1)(m+1)} = b_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{(m+n)1} = 0 & s_{(m+n)2} = 0 & \cdots & s_{(m+n)(m+1)} = b_{n-1} & b_{n-2} & \cdots & b_0 \end{pmatrix}$$

If A and B have integer coefficients with non-zero resultant r, then the polynomials equation

$$AU + BV = r$$

has a unique solution U, V such that degree(U) <degree(B) and degree(V) <degree(A), and this solution has integer coefficients.

Input :

```
resultant(x^3-p*x+q,3*x^2-p,x)
```

Output :

```
-4*p^3-27*q^2
```

Remark
discriminant(P)=resultant(P,P').

An example using the resultant
Let, $F1$ and $F2$ be 2 fixed points in the plane and A, a variable point on the circle of center $F1$ and radius $2a$. Find the cartesian equation of the set of points M, intersection of the line $F1A$ and of the perpendicular bisector of $F2A$.

Geometric answer :

$$MF1 + MF2 = MF1 + MA = F1A = 2a$$

hence M is on an ellipse with focus $F1, F2$ and major axis $2a$.

Analytic answer : In the Cartesian coordinate system with center $F1$ and x-axis having the same direction as the vector $F1F2$, the coordinates of A are :

$$A = (2a\cos(\theta), 2a\sin(\theta))$$

where θ is the (Ox, OA) angle. Now choose $t = \tan(\theta/2)$ as parameter, so that the coordinates of A are rational functions with respect to t. More precisely :

$$A = (ax, ay) = (2a\frac{1-t^2}{1+t^2}, 2a\frac{2t}{1+t^2})$$

If $F1F2 = 2c$ and if I is the midpoint of $AF2$, since the coordinates of $F2$ are $F2 = (2c, 0)$, the coordinates of I

$$I = (c + ax/2; ay/2) = (c + a\frac{1-t^2}{1+t^2}; a\frac{2t}{1+t^2})$$

IM is orthogonal to $AF2$, hence $M = (x; y)$ satisfies the equation $eq1 = 0$ where

$$eq1 := (x - ix) * (ax - 2 * c) + (y - iy) * ay$$

But $M = (x, y)$ is also on $F1A$, hence M satisfies the equation $eq2 = 0$

$$eq2 := y/x - ay/ax$$

The resultant of both equations with respect to t `resultant(eq1,eq2,t)` is a polynomial $eq3$ depending on the variables x, y, independent of t which is the cartesian equation of the set of points M when t varies.
Input :

```
ax:=2*a*(1-t^2)/(1+t^2);ay:=2*a*2*t/(1+t^2);
ix:=(ax+2*c)/2; iy:=(ay/2)
eq1:=(x-ix)*(ax-2*c)+(y-iy)*ay
eq2:=y/x-ay/ax
factor(resultant(eq1,eq2,t))
```

Output gives as resultant :

$$-(64\cdot(\mathtt{x}^2+\mathtt{y}^2)\cdot(\mathtt{x}^2\cdot\mathtt{a}^2-\mathtt{x}^2\cdot\mathtt{c}^2+-2\cdot\mathtt{x}\cdot\mathtt{a}^2\cdot\mathtt{c}+2\cdot\mathtt{x}\cdot\mathtt{c}^3-\mathtt{a}^4+2\cdot\mathtt{a}^2\cdot\mathtt{c}^2+\mathtt{a}^2\cdot\mathtt{y}^2-\mathtt{c}^4))$$

The factor $-64\cdot(\mathtt{x}^2+\mathtt{y}^2)$ is always different from zero, hence the locus equation of M :

$$\mathtt{x}^2\mathtt{a}^2-\mathtt{x}^2\mathtt{c}^2+-2\mathtt{x}\mathtt{a}^2\mathtt{c}+2\mathtt{x}\mathtt{c}^3-\mathtt{a}^4+2\mathtt{a}^2\mathtt{c}^2+\mathtt{a}^2\mathtt{y}^2-\mathtt{c}^4=0$$

If the frame origin is O, the middle point of $F1F2$, we find the cartesian equation of an ellipse. To make the change of origin $\overrightarrow{F1M}=\overrightarrow{F1O}+\overrightarrow{OM}$, input :

$$\mathtt{normal}(\mathtt{subst}(\mathtt{x}^2\cdot\mathtt{a}^2-\mathtt{x}^2\cdot\mathtt{c}^2+-2\cdot\mathtt{x}\cdot\mathtt{a}^2\cdot\mathtt{c}+2\cdot\mathtt{x}\cdot\mathtt{c}^3-\mathtt{a}^4+2\cdot\mathtt{a}^2\cdot\mathtt{c}^2+$$

$$\mathtt{a}^2\cdot\mathtt{y}^2-\mathtt{c}^4,[\mathtt{x},\mathtt{y}]=[\mathtt{c}+\mathtt{X},\mathtt{Y}]))$$

Output :

$$-\mathtt{c}^2*\mathtt{X}^2+\mathtt{c}^2*\mathtt{a}^2+\mathtt{X}^2*\mathtt{a}^2-\mathtt{a}^4+\mathtt{a}^2*\mathtt{Y}^2$$

or if $b^2=a^2-c^2$, input :

$$\mathtt{normal}(\mathtt{subst}(-\mathtt{c}^2*\mathtt{X}^2+\mathtt{c}^2*\mathtt{a}^2+\mathtt{X}^2*\mathtt{a}^2-\mathtt{a}^4+\mathtt{a}^2*\mathtt{Y}^2,\mathtt{c}^2=\mathtt{a}^2-\mathtt{b}^2))$$

Output :

$$-\mathtt{a}^2*\mathtt{b}^2+\mathtt{a}^2*\mathtt{Y}^2+\mathtt{b}^2*\mathtt{X}^2$$

that is to say, after division by a^2*b^2, M verifies the equation :

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

Another example using the resultant

Let $F1$ and $F2$ be fixed points and A a variable point on the circle of center $F1$ and radius $2a$. Find the cartesian equation of the hull of D, the segment bisector of $F2A$.

The segment bisector of $F2A$ is tangent to the ellipse of focus $F1, F2$ and major axis $2a$.

In the Cartesian coordinate system of center $F1$ and x-axis having the same direction than the vector $F1F2$, the coordinates of A are :

$$A=(2a\cos(\theta);2a\sin(\theta))$$

where θ is the (Ox,OA) angle. Choose $t=\tan(\theta/2)$ as parameter such that the coordinates of A are rational functions with respect to t. More precisely :

$$A=(ax;ay)=(2a\frac{1-t^2}{1+t^2};2a\frac{2t}{1+t^2})$$

If $F1F2=2c$ and if I is the middle point of $AF2$:

$$F2=(2c,0),\quad I=(c+ax/2;ay/2)=(c+a\frac{1-t^2}{1+t^2};a\frac{2t}{1+t^2})$$

Since D is orthogonal to $AF2$, the equation of D is $eq1 = 0$ where

$$eq1 := (x - ix) * (ax - 2 * c) + (y - iy) * ay$$

So, the hull of D is the locus of M, the intersection point of D and D' where D' has equation $eq2 := diff(eq1, t) = 0$. Input :

```
ax:=2*a*(1-t^2)/(1+t^2);ay:=2*a*2*t/(1+t^2);
ix:=(ax+2*c)/2; iy:=(ay/2)
eq1:=normal((x-ix)*(ax-2*c)+(y-iy)*ay)
eq2:=normal(diff(eq1,t))
factor(resultant(eq1,eq2,t))
```

Output gives as resultant :

$$(-(64 \cdot \mathtt{a\hat{}2})) \cdot (\mathtt{x\hat{}2} + \mathtt{y\hat{}2}) \cdot (\mathtt{x\hat{}2} \cdot \mathtt{a\hat{}2} - \mathtt{x\hat{}2} \cdot \mathtt{c\hat{}2} + -2 \cdot \mathtt{x} \cdot \mathtt{a\hat{}2} \cdot \mathtt{c} + 2 \cdot \mathtt{x} \cdot \mathtt{c\hat{}3} - \mathtt{a\hat{}4} + 2 \cdot \mathtt{a\hat{}2} \cdot \mathtt{c\hat{}2} + \mathtt{a\hat{}2} \cdot \mathtt{y\hat{}2} - \mathtt{c\hat{}4})$$

The factor $-64 \cdot (\mathtt{x}^2 + \mathtt{y}^2)$ is always different from zero, therefore the locus equation is :

$$\mathtt{x}^2\mathtt{a}^2 - \mathtt{x}^2\mathtt{c}^2 + -2\mathtt{x}\mathtt{a}^2\mathtt{c} + 2\mathtt{x}\mathtt{c}^3 - \mathtt{a}^4 + 2\mathtt{a}^2\mathtt{c}^2 + \mathtt{a}^2\mathtt{y}^2 - \mathtt{c}^4 = 0$$

If O, the middle point of $F1F2$, is chosen as origin, we find again the cartesian equation of the ellipse :

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

5.27 Orthogonal polynomials

5.27.1 Legendre polynomials: `legendre`

`legendre` takes as argument an integer n and optionally a variable name (by default x).
`legendre` returns the Legendre polynomial of degree n : it is a polynomial $L(n, x)$, solution of the differential equation:

$$(x^2 - 1)y'' - 2xy' - n(n + 1)y = 0$$

The Legendre polynomials verify the following recurrence relation:

$$L(0, x) = 1, \quad L(1, x) = x, \quad L(n, x) = \frac{2n - 1}{n} x L(n-1, x) - \frac{n - 1}{n} L(n-2, x)$$

These polynomials are orthogonal for the scalar product :

$$< f, g >= \int_{-1}^{+1} f(x)g(x)\, dx$$

Input :

```
legendre(4)
```

Output :

```
(35*x^4+-30*x^2+3)/8
```

Input :

```
legendre(4,y)
```

Output :

```
(35*y^4+-30*y^2+3)/8
```

5.27.2 Hermite polynomial : `hermite`

`hermite` takes as argument an integer n and optionally a variable name (by default x).
`hermite` returns the Hermite polynomial of degree n.
If $H(n,x)$ denotes the Hermite polynomial of degree n, the following recurrence relation holds:

$$H(0,x)=1,\quad H(1,x)=2x,\quad H(n,x)=2xH(n-1,x)-2(n-1)H(n-2,x)$$

These polynomials are orthogonal for the scalar product:

$$< f,g >= \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}dx$$

Input :

```
hermite(6)
```

Output :

```
64*x^6+-480*x^4+720*x^2-120
```

Input :

```
hermite(6,y)
```

Output :

```
64*y^6+-480*y^4+720*y^2-120
```

5.27.3 Laguerre polynomials: `laguerre`

`laguerre` takes as argument an integer n and optionally a variable name (by default x) and a parameter name (by default a).
`laguerre` returns the Laguerre polynomial of degree n and of parameter a.
If $L(n,a,x)$ denotes the Laguerre polynomial of degree n and parameter a, the following recurrence relation holds:

$$L(0,a,x)=1,\quad L(1,a,x)=1+a-x,\quad L(n,a,x)=\frac{2n+a-1-x}{n}L(n-1,a,x)-\frac{n+a-1}{n}L(n-$$

These polynomials are orthogonal for the scalar product

$$< f,g >= \int_{0}^{+\infty} f(x)g(x)x^a e^{-x}dx$$

Input :

```
laguerre(2)
```

Output :

```
(a^2+-2*a*x+3*a+x^2+-4*x+2)/2
```

Input :

```
laguerre(2,y)
```

Output :

```
(a^2+-2*a*y+3*a+y^2+-4*y+2)/2
```

Input :

```
laguerre(2,y,b)
```

Output :

```
(b^2+-2*b*y+3*b+y^2+-4*y+2)/2
```

5.27.4 Tchebychev polynomials of the first kind: `tchebyshev1`

`tchebyshev1` takes as argument an integer n and optionally a variable name (by default x).
`tchebyshev1` returns the Tchebychev polynomial of first kind of degree n.
The Tchebychev polynomial of first kind $T(n, x)$ is defined by

$$T(n, x) = \cos(n \arccos(x))$$

and satisfy the recurrence relation:

$$T(0, x) = 1, \quad T(1, x) = x, \quad T(n, x) = 2xT(n-1, x) - T(n-2, x)$$

The polynomials $T(n, x)$ are orthogonal for the scalar product

$$< f, g >= \int_{-1}^{+1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$$

Input :

```
tchebyshev1(4)
```

Output :

```
8*x^4+-8*x^2+1
```

Input :

```
tchebyshev1(4,y)
```

Output :

```
8*y^4+-8*y^2+1
```

Indeed

$$\begin{aligned}
\cos(4x) &= Re((\cos(x) + i\sin(x))^4) \\
&= \cos(x)^4 - 6.\cos(x)^2(1-\cos(x)^2) + ((1-\cos(x)^2)^2 \\
&= T(4, \cos(x))
\end{aligned}$$

5.27.5 Tchebychev polynomial of the second kind: `tchebyshev2`

`tchebyshev2` takes as argument an integer n and optionally a variable name (by default x).
`tchebyshev2` returns the Tchebychev polynomial of second kind of degree n.
The Tchebychev polynomial of second kind $U(n,x)$ is defined by:

$$U(n,x) = \frac{\sin((n+1).\arccos(x))}{\sin(\arccos(x))}$$

or equivalently:

$$\sin((n+1)x) = \sin(x) * U(n, \cos(x))$$

Then $U(n,x)$ satisfies the recurrence relation:

$$U(0,x) = 1, \quad U(1,x) = 2x, \quad U(n,x) = 2xU(n-1,x) - U(n-2,x)$$

The polynomials $U(n,x)$ are orthogonal for the scalar product

$$< f, g >= \int_{-1}^{+1} f(x)g(x)\sqrt{1-x^2}dx$$

Input :

```
tchebyshev2(3)
```

Output :

```
8*x^3+-4*x
```

Input :

```
tchebyshev2(3,y)
```

Output :

```
8*y^3+-4*y
```

Indeed:

$$\sin(4x) = \sin(x) * (8 * \cos(x)^3 - 4\cos(x)) = \sin(x) * U(3, \cos(x))$$

5.28 Gröbner basis and Gröbner reduction

5.28.1 Gröbner basis : `gbasis`

`gbasis` takes at least two arguments

- a vector of multivariate polynomials
- a vector of variables names,

Optional arguments may be used to specify the ordering and algorithms. By default, the ordering is lexicographic (with respect to the list of variable names ordering) and the polynomials are written in decreasing power orders with respect to this order. For example, the output will be like ... $+ x^2y^4z^3 + x^2y^3z^4 +$ ... if the second argument is $[x, y, z]$ because $(2, 4, 3) > (2, 3, 4)$ but the output would be like ... $+ x^2y^3z^4 + x^2y^4z^3 +$ ... if the second argument is $[x, z, y]$.
`gbasis` returns a Gröbner basis of the polynomial ideal spanned by these polynomials.

Property

If I is an ideal and if $(G_k)_{k\in K}$ is a Gröbner basis of this ideal I then, if F is a non-zero polynomial in I, the greatest monomial of F is divisible by the greatest monomial of one of the G_k. In other words, if you do an euclidean division of $F \neq 0$ by the corresponding G_k, take the remainder of this division, do again the same and so on, at some point you get a null remainder.

Input :

```
gbasis([2*x*y-y^2,x^2-2*x*y],[x,y])
```

Output :

```
[4*x^2+-4*y^2,2*x*y-y^2,-(3*y^3)]
```

As indicated above, `gbasis` may have more than 2 arguments :

- `plex` (lexicographic only), `tdeg` (total degree then lexicographic order), `revlex` (total degree then inverse lexicographic order), to specify an order on the monomials (`plex` is the order by default),
- `with_cocoa=true` or `with_cocoa=false`, if you want to use the `CoCoA` library to compute the Gröbner basis (recommended, requires that `CoCoA` support compiled in)
- `with_f5=true` or `with_f5=false` for using the F5 algorithm of the `CoCoA` library . In this case the specified order is not used (the polynomials are homogenized).

Input :

```
gbasis([x1+x2+x3,x1*x2+x1*x3+x2*x3,x1*x2*x3-1],
        [x1,x2,x3],tdeg,with_cocoa=false)
```

Output

```
[x3^3-1,-x2^2-x2*x3-x3^2,x1+x2+x3]
```

5.28.2 Gröbner reduction : `greduce`

`greduce` has three arguments : a multivariate polynomial, a vector made of polynomials which is supposed to be a Gröbner basis, and a vector of variable names.
`greduce` returns the reduction of the polynomial given as first argument with respect to the Gröbner basis given as the second argument. It is 0 if and only if the polynomial belongs to the ideal.

Input :

```
greduce(x*y-1,[x^2-y^2,2*x*y-y^2,y^3],[x,y])
```

Output :

```
y^2-2
```

that is to say $xy - 1 = \frac{1}{2}(y^2 - 2) \mod I$ where I is the ideal generated by the Gröbner basis $[x^2 - y^2, 2xy - y^2, y^3]$, because $y^2 - 2$ is the euclidean division remainder of $2(xy - 1)$ by $G_2 = 2xy - y^2$.

Like `gbasis` (cf. 5.28.1), `greduce` may have more than 3 arguments to specify ordering and algorithm if they differ from the default (lexicographic ordering).
Input :

```
greduce(x1^2*x3^2,[x3^3-1,-x2^2-x2*x3-x3^2,x1+x2+x3],
                    [x1,x2,x3],tdeg)
```

Output

```
x2
```

5.28.3 Build a polynomial from its evaluation : `genpoly`

`genpoly` takes three arguments : a polynomial P with $n-1$ variables, an integer b and the name of a variable `var`.
`genpoly` returns the polynomial Q with n variables (the P variables and the variable `var` given as second argument), such that :

- `subst(Q,var=b)==P`
- the coefficients of Q belongs to the interval $(-b/2\ ,\ b/2]$

In other words, P is written in base b but using the convention that the euclidean remainder belongs to $]-b/2\ ;\ b/2]$ (this convention is also known as s-mod representation). Input :

```
genpoly(61,6,x)
```

Output :

```
2*x^2-2*x+1
```

Indeed 61 divided by 6 is 10 with remainder 1, then 10 divided by 6 is 2 with remainder -2 (instead of the usual quotient 1 and remainder 4 out of bounds),

$$61 = 2 * 6^2 - 2 * 6 + 1$$

Input :

```
genpoly(5,6,x)
```

Output :

```
x-1
```

Indeed : $5 = 6 - 1$
Input :

```
genpoly(7,6,x)
```

Output :

```
x+1
```

Indeed : $7 = 6 + 1$
Input :

```
genpoly(7*y+5,6,x)
```

Output :

```
x*y+x+y-1
```

Indeed : $x * y + x + y - 1 = y(x + 1) + (x - 1)$
Input :

```
genpoly(7*y+5*z^2,6,x)
```

Output :

```
x*y+x*z+y-z
```

Indeed : $x * y + x * z + y - z = y * (x + 1) + z * (x - 1)$

5.29 Rational fractions

5.29.1 Numerator : `getNum`

`getNum` takes as argument a rational fraction and returns the numerator of this fraction. Unlike `numer`, `getNum` does not simplify the fraction before extracting the numerator.
Input :

```
getNum((x^2-1)/(x-1))
```

Output :

```
x^2-1
```

Input :

```
getNum((x^2+2*x+1)/(x^2-1))
```

Output :

```
x^2+2*x+1
```

5.29.2 Numerator after simplification : `numer`

`numer` takes as argument a rational fraction and returns the numerator of the irreducible representation of this fraction (see also 5.8.3).
Input :

```
numer((x^2-1)/(x-1))
```

Output :

```
x+1
```

Input :

```
numer((x^2+2*x+1)/(x^2-1))
```

Output :

```
x+1
```

5.29.3 Denominator : `getDenom`

`getDenom` takes as argument a rational fraction and returns the denominator of this fraction. Unlike `denom`, `getDenom` does not simplify the fraction before extracting the denominator.
Input :

```
getDenom((x^2-1)/(x-1))
```

Output :

```
x-1
```

Input :

```
getDenom((x^2+2*x+1)/(x^2-1))
```

Output :

```
x^2-1
```

5.29.4 Denominator after simplification : `denom`

`denom` (or `getDenom`) takes as argument a rational fraction and returns the denominator of an irreducible representation of this fraction (see also 5.8.4).
Input :

```
denom((x^2-1)/(x-1))
```

Output :

```
1
```

Input :

```
denom((x^2+2*x+1)/(x^2-1))
```

Output :

```
x-1
```

5.29.5 Numerator and denominator : `f2nd fxnd`

`f2nd` (or `fxnd`) takes as argument a rational fraction and returns the list of the numerator and the denominator of the irreducible representation of this fraction (see also 5.8.5).
Input :

```
f2nd((x^2-1)/(x-1))
```

Output :

```
[x+1,1]
```

Input :

```
f2nd((x^2+2*x+1)/(x^2-1))
```

Output :

```
[x+1,x-1]
```

5.29.6 Simplify : `simp2`

`simp2` takes as argument two polynomials (or two integers see 5.8.6). These two polynomials are seen as the numerator and denominator of a rational fraction.
`simp2` returns a list of two polynomials seen as the numerator and denominator of the irreducible representation of this rational fraction.
Input :

```
simp2(x^3-1,x^2-1)
```

Output :

```
[x^2+x+1,x+1]
```

5.29.7 Common denominator : `comDenom`

`comDenom` takes as argument a sum of rational fractions.
`comDenom` rewrite the sum as a unique rational fraction. The denominator of this rational fraction is the common denominator of the rational fractions given as argument.
Input :

```
comDenom(x-1/(x-1)-1/(x^2-1))
```

Output :

```
(x^3+-2*x-2)/(x^2-1)
```

5.29.8 Integer and fractional part : `propfrac`

`propfrac` takes as argument a rational fraction.
`propfrac` rewrites this rational fraction as the sum of its integer part and proper fractional part.
`propfrac(A(x)/B(x))` writes the fraction $\frac{A(x)}{B(x)}$ (after reduction), as :

$$Q(x) + \frac{R(x)}{B(x)} \quad \text{where } R(x) = 0 \text{ or } 0 \leq \text{degree}(R(x)) < \text{degree}(B(x))$$

Input :

```
propfrac((5*x+3)*(x-1)/(x+2))
```

Output :

```
5*x-12+21/(x+2)
```

5.29.9 Partial fraction expansion : `partfrac`

`partfrac` takes as argument a rational fraction.
`partfrac` returns the partial fraction expansion of this rational fraction.
The `partfrac` command is equivalent to the `convert` command with `parfrac` (or `partfrac` or `fullparfrac`) as option (see also 5.21.24).
Example :
Find the partial fraction expansion of :

$$\frac{x^5 - 2x^3 + 1}{x^4 - 2x^3 + 2x^2 - 2x + 1}$$

Input :

```
partfrac((x^5-2*x^3+1)/(x^4-2*x^3+2*x^2-2*x+1))
```

Output in real mode :

```
x+2-1/(2*(x-1))+(x-3)/(2*(x^2+1))
```

Output in complex mode:

```
x+2+(-1+2*i)/((2-2*i)*((i)*x+1))+1/(2*(-x+1))+
(-1-2*i)/((2-2*i)*(x+i))
```

5.30 Exact roots of a polynomial

5.30.1 Exact bounds for complex roots of a polynomial : `complexroot`

`complexroot` takes 2 or 4 arguments : a polynomial and a real number ϵ and optionally two complex numbers α, β.
`complexroot` returns a list of vectors.

- If `complexroot` has 2 arguments, the elements of each vector are

- either an interval (the boundaries of this interval are the opposite vertices of a rectangle with sides parallel to the axis and containing a complex root of the polynomial) and the multiplicity of this root.
  Let the interval be $[a_1 + ib_1, a_2 + ib_2]$ then $|a_1 - a_2| < \epsilon$, $|b_1 - b_2| < \epsilon$ and the root $a + ib$ verifies $a_1 \leq a \leq a_2$ and $b_1 \leq b \leq b_2$.
- or the value of an exact complex root of the polynomial and the multiplicity of this root

• If `complexroot` has 4 arguments, `complexroot` returns a list of vectors as above, but only for the roots lying in the rectangle with sides parallel to the axis having α, β as opposite vertices.

To find the roots of $x^3 + 1$, input:

```
complexroot(x^3+1,0.1)
```

Output :

```
[[-1,1],[[(4-7*i)/8,(8-13*i)/16],1],[[(8+13*i)/16,(4+7*i)/8],1]]
```

Hence, for $x^3 + 1$:

• -1 is a root of multiplicity 1,

• 1/2+i*b is a root of multiplicity 1 with $-7/8 \leq b \leq -13/16$,

• 1/2+i*c is a root of multiplicity 1 with $13/16 \leq c \leq 7/8$.

To find the roots of x^3+1 lying inside the rectangle of opposite vertices $-1, 1+2*i$, input:

```
complexroot(x^3+1,0.1,-1,1+2*i)
```

Output :

```
[[-1,1],[[(8+13*i)/16,(4+7*i)/8],1]]
```

5.30.2 Exact bounds for real roots of a polynomial : `realroot`

`realroot` has 2 or 4 arguments : a polynomial and a real number ϵ and optionally two reals numbers α, β.
`realroot` returns a list of vectors.

• If `realroot` has 2 arguments, the elements of each vector are

- either a real interval containing a real root of the polynomial and the multiplicity of this root. Let the interval be $[a_1, a_2]$ then $|a_1 - a_2| < \epsilon$ and the root a verifies $a_1 \leq a \leq a_2$.
- or the value of an exact real root of the polynomial and the multiplicity of this root.

- If `realroot` has 4 arguments, `realroot` returns a list of vectors as above, but only for the roots inside the interval $[\alpha, \beta]$.

To find the real roots of $x^3 + 1$, input:

```
realroot(x^3+1, 0.1)
```

Output :

```
[[-1,1]]
```

To find the real roots of $x^3 - x^2 - 2x + 2$, input:

```
realroot(x^3-x^2-2*x+2, 0.1)
```

Output :

```
[[1,1],[[(-3)/2,(-45)/32],1],[[45/32,3/2],1]]
```

To find the real roots of $x^3 - x^2 - 2x + 2$ in the interval $[0; 2]$, input:

```
realroot(x^3-x^2-2*x+2, 0.1,0,2)
```

Output :

```
[[1,1],[[11/8,23/16],1]]
```

5.30.3 Exact values of rational roots of a polynomial : `rationalroot`

`rationalroot` takes 1 or 3 arguments : a polynomial and optionally two real numbers α, β.

- If `rationalroot` has 1 argument, `rationalroot` returns the list of the value of the rational roots of the polynomial without multiplicity.
- If `rationalroot` has 3 arguments, `rationalroot` returns only the rational roots of the polynomial which are in the interval $[\alpha, \beta]$.

To find the rational roots of $2 * x^3 - 3 * x^2 - 8 * x + 12$, input:

```
rationalroot(2*x^3-3*x^2-8*x+12)
```

Output :

```
[2,3/2,-2]
```

To find the rational roots of $2 * x^3 - 3 * x^2 - 8 * x + 12$ in $[1; 2]$, input:

```
rationalroot(2*x^3-3*x^2-8*x+12,1,2)
```

Output :

```
[2,3/2]
```

To find the rational roots of $2 * x^3 - 3 * x^2 + 8 * x - 12$, input:

```
rationalroot(2*x^3-3*x^2+8*x-12)
```

Output :

```
[3/2]
```

To find the rational roots of $2 * x^3 - 3 * x^2 + 8 * x - 12$, input:

```
rationalroot(2*x^3-3*x^2+8*x-12)
```

Output :

```
[3/2]
```

To find the rational roots of $(3 * x - 2)^2 * (2x + 1) = 18 * x^3 - 15 * x^2 - 4 * x + 4$, input:

```
rationalroot(18*x^3-15*x^2-4*x+4)
```

Output :

```
[(-1)/2,2/3]
```

5.30.4 Exact values of the complex rational roots of a polynomial : `crationalroot`

`crationalroot` takes 1 or 3 arguments : a polynomial and optionally two complex numbers α, β.

- If `crationalroot` has 1 argument, `crationalroot` returns the list of the complex rational roots of the polynomial without multiplicity.
- if `crationalroot` has 3 arguments, `crationalroot` returns only the complex rational roots of the polynomial which are in the rectangle with sides parallel to the axis having $[\alpha, \beta]$ as opposite vertices.

To find the rational complex roots of $(x^2+4)*(2x-3) = 2*x^3-3*x^2+8*x-12$, input :

```
crationalroot(2*x^3-3*x^2+8*x-12)
```

Output :

```
[2*i,3/2,-2*i]
```

5.31 Exact roots and poles

5.31.1 Roots and poles of a rational function : `froot`

`froot` takes a rational function $F(x)$ as argument.
`froot` returns a vector whose components are the roots and the poles of $F[x]$, each one followed by its multiplicity.
If `Xcas` can not find the exact values of the roots or poles, it tries to find approximate values if $F(x)$ has numeric coefficients.
Input :

```
froot((x^5-2*x^4+x^3)/(x-2))
```

Output :

```
[1,2,0,3,2,-1]
```

Hence, for $F(x) = \dfrac{x^5 - 2.x^4 + x^3}{x - 2}$:

- 1 is a root of multiplicity 2,
- 0 is a root of multiplicity 3,
- 2 is a pole of order 1.

Input :

```
froot((x^3-2*x^2+1)/(x-2))
```

Output :

```
[1,1,(1+sqrt(5))/2,1,(1-sqrt(5))/2,1,2,-1]
```

Remark : to have the complex roots and poles, check `Complex` in the `cas` configuration (red button giving the state line).
Input :

```
froot((x^2+1)/(x-2))
```

Output :

```
[-i,1,i,1,2,-1]
```

5.31.2 Rational function given by roots and poles : `fcoeff`

`fcoeff` has as argument a vector whose components are the roots and poles of a rational function $F[x]$, each one followed by its multiplicity.
`fcoeff` returns the rational function $F(x)$.
Input :

```
fcoeff([1,2,0,3,2,-1])
```

Output :

```
(x-1)^2*x^3/(x-2)
```

5.32 Computing in $\mathbb{Z}/p\mathbb{Z}$ or in $\mathbb{Z}/p\mathbb{Z}[x]$

The way to compute over $\mathbb{Z}/p\mathbb{Z}$ or over $\mathbb{Z}/p\mathbb{Z}[x]$ depends on the syntax mode :

- In `Xcas` mode, an object n over $\mathbb{Z}/p\mathbb{Z}$ is written $n\%p$. Some examples of input for
  - an integer `n` in $\mathbb{Z}/13\mathbb{Z}$
    `n:=12%13`.

 - a vector `V` in $\mathbb{Z}/13\mathbb{Z}$
   `V:=[1,2,3]%13` or `V:=[1%13,2%13,3%13]`.
 - a matrix `A` in $\mathbb{Z}/13\mathbb{Z}$
   `A:=[[1,2,3],[2,3,4]]%13` or
   `A:=[[1%13,2%13,3%13],[[2%13,3%13,4%13]]`.
 - a polynomial `A` in $\mathbb{Z}/13\mathbb{Z}[x]$ in symbolic representation
   `A:=(2*x^2+3*x-1)%13` or
   `A:=2%13*x^2+3%13*x-1%13`.
 - a polynomial `A` in $\mathbb{Z}/13\mathbb{Z}[x]$ in list representation
   `A:=poly1[1,2,3]%13` or `A:=poly1[1%13,2%13,3%13]`.

 To recover an object `o` with integer coefficients instead of modular coefficients, input `o % 0`. For example, input `o:=4%7` and `o%0`,then output is `-3`.

- In `Maple` mode, integers modulo p are represented like usual integers instead of using specific modular integers. To avoid confusion with normal commands, modular commands are written with a capital letter (inert form) and followed by the mod command (see also the next section).

Remark

- For some commands in $\mathbb{Z}/p\mathbb{Z}$ or in $\mathbb{Z}/p\mathbb{Z}[x]$, `p` must be a prime integer.
- The representation is the symmetric representation :
  `11%13` returns `-2%13`.

5.32.1 Expand and reduce : `normal`

`normal` takes as argument a polynomial expression.
`normal` expands and reduces this expression in $\mathbb{Z}/p\mathbb{Z}[x]$.
Input :

```
normal(((2*x^2+12)*( 5*x-4))%13)
```

Output :

```
(-3%13)*x^3+(5%13)*x^2+(-5%13)*x+4%13
```

5.32.2 Addition in $\mathbb{Z}/p\mathbb{Z}$ or in $\mathbb{Z}/p\mathbb{Z}[x]$: +

`+` adds two integers in $\mathbb{Z}/p\mathbb{Z}$, or two polynomials in $\mathbb{Z}/p\mathbb{Z}[x]$. For polynomial expressions, use the `normal` command to simplify.
For integers in $\mathbb{Z}/p\mathbb{Z}$, input :

```
3%13+10%13
```

Output :

```
0%13
```

For polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$, input :

```
normal((11*x+5 )% 13+(8*x+6)%13)
```

or

```
normal(11% 13*x+5%13+8% 13*x+6%13)
```

Output :

```
(6%13)*x+-2%13
```

5.32.3 Subtraction in $\mathbb{Z}/p\mathbb{Z}$ or in $\mathbb{Z}/p\mathbb{Z}[x]$: -

- subtracts two integers in $\mathbb{Z}/p\mathbb{Z}$ or two polynomials in $\mathbb{Z}/p\mathbb{Z}[x]$. For polynomial expressions, use the `normal` command to simplify.
For integers in $\mathbb{Z}/p\mathbb{Z}$, input :

```
31%13-10%13
```

Output :

```
-5%13
```

For polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$, input :

```
normal((11*x+5)%13-(8*x+6)%13)
```

or :

```
normal(11% 13*x+5%13-8% 13*x+6%13)
```

Output :

```
(3%13)*x+-1%13
```

5.32.4 Multiplication in $\mathbb{Z}/p\mathbb{Z}$ or in $\mathbb{Z}/p\mathbb{Z}[x]$: *

* multiplies two integers in $\mathbb{Z}/p\mathbb{Z}$ or two polynomials in $\mathbb{Z}/p\mathbb{Z}[x]$. For polynomial expressions, use the `normal` command to simplify.
For integers in $\mathbb{Z}/p\mathbb{Z}$, input :

```
31%13*10%13
```

Output :

```
-2%13
```

For polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$, input :

```
normal((11*x+5)%13*(8*x+6 )% 13)
```

or :

```
normal((11% 13*x+5%13)*(8% 13*x+6%13))
```

Output :

```
(-3%13)*x^2+(2%13)*x+4%13
```

5.32.5 Euclidean quotient : quo

quo takes as arguments two polynomials A and B with coefficients in $\mathbb{Z}/p\mathbb{Z}$, where A and B are list polynomials or symbolic polynomials with respect to x or to an optional third argument.
quo returns the quotient of the euclidean division of A by B in $\mathbb{Z}/p\mathbb{Z}[x]$.
Input :

```
quo((x^3+x^2+1)%13,(2*x^2+4)%13)
```

or :

```
quo((x^3+x^2+1,2*x^2+4)%13)
```

Output:

```
(-6%13)*x+-6%13
```

Indeed $x^3+x^2+1 = (2x^2+4)(\frac{x+1}{2})+\frac{5x-4}{4}$ and $-3*4 = -6*2 = 1 \mod 13$.

5.32.6 Euclidean remainder : rem

rem takes as arguments two polynomials A and B with coefficients in $\mathbb{Z}/p\mathbb{Z}$, where A and B are list polynomials or symbolic polynomials with respect to x or to an optional third argument.
rem returns the remainder of the euclidean division of A by B in $\mathbb{Z}/p\mathbb{Z}[x]$.
Input :

```
rem((x^3+x^2+1)%13,(2*x^2+4)%13)
```

or :

```
rem((x^3+x^2+1,2*x^2+4)%13)
```

Output:

```
(-2%13)*x+-1%13
```

Indeed $x^3+x^2+1 = (2x^2+4)(\frac{x+1}{2})+\frac{5x-4}{4}$ and $-3*4 = -6*2 = 1 \mod 13$.

5.32.7 Euclidean quotient and euclidean remainder : quorem

quorem takes as arguments two polynomials A and B with coefficients in $\mathbb{Z}/p\mathbb{Z}$, where A and B are list polynomials or symbolic polynomials with respect to x or to an optional third argument.
quorem returns the list of the quotient and remainder of the euclidean division of A by B in $\mathbb{Z}/p\mathbb{Z}[x]$ (see also 5.6.12 and 5.26.6).
Input :

```
quorem((x^3+x^2+1)%13,(2*x^2+4)%13)
```

or :

```
quorem((x^3+x^2+1,2*x^2+4)%13)
```

Output:

```
[(-6%13)*x+-6%13,(-2%13)*x+-1%13]
```

Indeed $x^3 + x^2 + 1 = (2x^2 + 4)(\frac{x+1}{2}) + \frac{5x-4}{4}$
and $-3 * 4 = -6 * 2 = 1 \mod 13$.

5.32.8 Division in $\mathbb{Z}/p\mathbb{Z}$ or in $\mathbb{Z}/p\mathbb{Z}[x]$: /

/ divides two integers in $\mathbb{Z}/p\mathbb{Z}$ or two polynomials A and B in $\mathbb{Z}/p\mathbb{Z}[x]$.
For polynomials, the result is the irreducible representation of the fraction $\frac{A}{B}$ in $\mathbb{Z}/p\mathbb{Z}[x]$.
For integers in $\mathbb{Z}/p\mathbb{Z}$, input :

```
5%13/2% 13
```

Since 2 is invertible in $Z/13\mathbb{Z}$, we get the output :

```
-4%13
```

For polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$, input :

```
(2*x^2+5)%13/(5*x^2+2*x-3)%13
```

Output :

```
((6%13)*x+1%13)/((2%13)*x+2%13)
```

5.32.9 Power in $\mathbb{Z}/p\mathbb{Z}$ and in $\mathbb{Z}/p\mathbb{Z}[x]$: ^

To compute `a` to the power `n` in $\mathbb{Z}/p\mathbb{Z}$, we use the operator `^`. `Xcas` implementation is the binary power algorithm.
Input :

```
(5%13)^2
```

Output :

```
-1%13
```

To compute `A` to the power `n` in $\mathbb{Z}/p\mathbb{Z}[x]$, we use the operator ^ and the `normal` command .
Input :

```
normal(((2*x+1)%13)^5)
```

Output :

```
(6%13)*x^5+(2%13)*x^4+(2%13)*x^3+(1%13)*x^2+(-3%13)*x+1%13
```

because $10 = -3 \pmod{13}$, $40 = 1 \pmod{13}$, $80 = 2 \pmod{13}$, $32 = 6 \pmod{13}$.

5.32.10 Compute $a^n \mod p$: powmod powermod

powmod (or powermod) takes as argument a, n, p.
powmod (or powermod) returns $a^n \mod p$ in $[0; p-1]$.
Input :

```
powmod(5,2,13)
```

Output :

```
12
```

Input :

```
powmod(5,2,12)
```

Output :

```
1
```

5.32.11 Inverse in $\mathbb{Z}/p\mathbb{Z}$: inv inverse or /

To compute the inverse of an integer n in $\mathbb{Z}/p\mathbb{Z}$, input 1/n%p or inv(n%p) or inverse(n%p).
Input :

```
inv(3%13)
```

Output :

```
-4%13
```

Indeed $3 \times -4 = -12 = 1 \pmod{13}$.

5.32.12 Rebuild a fraction from its value modulo p : fracmod

fracmod takes two arguments, an integer n (representing a fraction) and an integer p (the modulus).
If possible, fracmod returns a fraction a/b such that

$$-\frac{\sqrt{p}}{2} < a \leq \frac{\sqrt{p}}{2}, \quad 0 \leq b < \frac{\sqrt{p}}{2}, \quad n \times b = a \pmod{p}$$

In other words $n = a/b \pmod{p}$.
Input :

```
fracmod(3,13)
```

Output :

```
-1/4
```

Indeed : $3 * -4 = -12 = 1 \pmod{13}$, hence $3 = -1/4\%13$.
Input :

```
fracmod(13,121)
```

Output :

```
-4/9
```

Indeed : $13 \times -9 = -117 = 4 \pmod{121}$ hence $13 = -4/9\%13$.

5.32.13 GCD in $\mathbb{Z}/p\mathbb{Z}[x]$: `gcd`

`gcd` takes as arguments two polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$ (p must be prime).
`gcd` returns the GCD of these polynomials computed in $\mathbb{Z}/p\mathbb{Z}[x]$ (see also 5.26.7 for polynomials with non modular coefficients).
Input :

```
gcd((2*x^2+5)%13,(5*x^2+2*x-3)%13)
```

Output :

```
(-4%13)*x+5%13
```

Input :

```
gcd((x^2+2*x+1,x^2-1)) mod 5)
```

Output :

```
x%5
```

Note the difference with a gcd computation in $\mathbb{Z}[X]$ followed by a reduction modulo 5, input:

```
gcd(x^2+2*x+1,x^2-1) mod 5
```

Output :

```
1
```

5.32.14 Factorization over $\mathbb{Z}/p\mathbb{Z}[x]$: `factor factoriser`

`factor` takes as argument a polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}[x]$.
`factor` factorizes this polynomial in $\mathbb{Z}/p\mathbb{Z}[x]$ (p must be prime).
Input :

```
factor((-3*x^3+5*x^2-5*x+4)%13)
```

Output :

```
((1%13)*x+-6%13)*((-3%13)*x^2+-5%13)
```

5.32.15 Determinant of a matrix in $\mathbb{Z}/p\mathbb{Z}$: `det`

`det` takes as argument a matrix A with coefficients in Z/pZ.
`det` returns the determinant of this matrix A.
Computations are done in $\mathbb{Z}/p\mathbb{Z}$ by Gauss reduction.
Input :

```
det([[1,2,9]%13,[3,10,0]%13,[3,11,1]%13])
```

or :

```
det([[1,2,9],[3,10,0],[3,11,1]]%13)
```

Output :

```
5%13
```

hence, in $\mathbb{Z}/13\mathbb{Z}$, the determinant of $A = [[1, 2, 9], [3, 10, 0], [3, 11, 1]]$ is `5%13` (in $\mathbb{Z}$, `det(A)=31`).

5.32.16 Inverse of a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$: `inv inverse`

`inverse` (or `inv`) takes as argument a matrix A in $\mathbb{Z}/p\mathbb{Z}$.
`inverse` (or `inv`) returns the inverse of the matrix A in $Z/p\mathbb{Z}$.
Input :

```
inverse([[1,2,9]%13,[3,10,0]%13,[3,11,1]%13])
```

or :

```
inv([[1,2,9]%13,[3,10,0]%13,[3,11,1]%13])
```

or :

```
inverse([[1,2,9],[3,10,0],[3,11,1]]%13)
```

or :

```
inv([[1,2,9],[3,10,0],[3,11,1]]%13)
```

Output :

```
[[2%13,-4%13,-5%13],[2%13,0%13,-5%13],
        [-2%13,-1%13,6%13]]
```

it is the inverse of $A = [[1, 2, 9], [3, 10, 0], [3, 11, 1]]$ in $\mathbb{Z}/13\mathbb{Z}$.

5.32.17 Row reduction to echelon form in $\mathbb{Z}/p\mathbb{Z}$: `rref`

`rref` finds the row reduction to echelon form of a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$.

This may be used to solve a linear system of equations with coefficients in $\mathbb{Z}/p\mathbb{Z}$ by rewriting it in matrix form (see also 5.54.3) :

```
A*X=B
```

`rref` takes as argument the augmented matrix of the system (the matrix obtained by augmenting matrix `A` to the right with the column vector `B`).
`rref` returns a matrix `[A1,B1]` : `A1` has 1 on its principal diagonal, and zeros outside, and the solutions in $\mathbb{Z}/p\mathbb{Z}$, of :

```
A1*X=B1
```

are the same as the solutions of:

```
A*X=B
```

Example, solve in $\mathbb{Z}/13\mathbb{Z}$

$$\begin{cases} x + 2\cdot y = 9 \\ 3\cdot x + 10\cdot y = 0 \end{cases}$$

Input :

```
rref([[1, 2, 9]%13,[3,10,0]%13])
```

or :

```
rref([[1, 2, 9],[3,10,0]])%13
```

Output :

```
[[1%13,0%13,3%13],[0%13,1%13,3%13]]
```

hence `x=3%13` and `y=3%13`.

5.32.18 Construction of a Galois field : `GF`

`GF` takes as arguments a prime integer p and an integer $n > 1$.
`GF` returns a Galois field of characteristic p having p^n elements.
Elements of the field and the field itself are represented by `GF(...)` where `...` is the following sequence:

- the characteristic p ($px = 0$),
- an irreducible primitive minimal polynomial generating an ideal I in $\mathbb{Z}/p\mathbb{Z}[X]$, the Galois field being the quotient of $\mathbb{Z}/p\mathbb{Z}[X]$ by I,
- the name of the polynomial variable, by default `x`,
- a polynomial (a remainder modulo the minimal polynomial) for an element of the field (field elements are represented with the additive representation) or `undef` for the field itself.

You should give a name to this field (for example `G:=GF(p,n)`), in order to build elements of the field from a polynomial in $\mathbb{Z}/p\mathbb{Z}[X]$, for example `G(x^3+x)`. Note that `G(x)` is a generator of the multiplicative group G^*.
Input :

```
G:=GF(2,8)
```

Output :

```
GF(2,x^8-x^6-x^4-x^3-x^2-x-1,x,undef)
```

The field G has $2^8 = 256$ elements and x generates the multiplicative group of this field ($\{1, x, x^2, ...x^{254}\}$).
Input :

```
G(x^9)
```

Output :

```
GF(2,x^8-x^6-x^4-x^3-x^2-x-1,x,x^7+x^5+x^4+x^3+x^2+x)
```

indeed $x^8 = x^6 + x^4 + x^3 + x^2 + x + 1$, hence $x^9 = x^7 + x^5 + x^4 + x^3 + x^2 + x$.
Input :

```
G(x)^255
```

Output should be the unit, indeed:

```
GF(2,x^8-x^6-x^4-x^3-x^2-x-1,x,1)
```

As one can see in these examples, the output contains many times the same information that you would prefer not to see if you work many times with the same field. For this reason, the definition of a Galois field may have an optional argument, a variable name which will be used thereafter to represent elements of the field. Since you will also most likely want to modify the name of the indeterminate, the field name is grouped with the variable name in a list passed as third argument to `GF`. Note that these two variable names must be quoted.
Example,
Input :

```
G:=GF(2,2,['w','G']):; G(w^2)
```

Output :

```
Done, G(w+1)
```

Input :

```
G(w^3)
```

Output :

```
G(1)
```

Hence, the elements of `GF(2,2)` are `G(0),G(1),G(w),G(w^2)=G(w+1)`.

We may also impose the irreducible primitive polynomial that we wish to use, by putting it as second argument (instead of n), for example :

```
G:=GF(2,w^8+w^6+w^3+w^2+1,['w','G'])
```

If the polynomial is not primitive, `Xcas` will replace it automatically by a primitive polynomial, for example :
Input :

```
G:=GF(2,w^8+w^7+w^5+w+1,['w','G'])
```

Output :

```
G:=GF(2,w^8-w^6-w^3-w^2-1,['w','G'],undef)
```

5.32.19 Factorize a polynomial with coefficients in a Galois field : `factor`

`factor` can also factorize a univariate polynomial with coefficients in a Galois field.
Input for example to have G=$\mathbb{F}_4$:

```
G:=GF(2,2,['w','G'])
```

Output :

```
GF(2,w^2+w+1,[w,G],undef)
```

Input for example :

```
a:=G(w)
```

```
factor(a^2*x^2+1)
```

Output :

```
(G(w+1))*(x+G(w+1))^2
```

5.33 Compute in $\mathbb{Z}/p\mathbb{Z}[x]$ using Maple syntax

5.33.1 Euclidean quotient : Quo

Quo is the inert form of quo.
Quo returns the euclidean quotient between two polynomials without evaluation. It is used in conjunction with mod in Maple syntax mode to compute the euclidean quotient of the division of two polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$.
Input in Xcas mode:

```
Quo((x^3+x^2+1) mod 13,(2*x^2+4) mod 13)
```

Output :

```
quo((x^3+x^2+1)%13,(2*x^2+4)%13)
```

you need to eval(ans()) to get :

```
(-6%13)*x+-6%13
```

Input in Maple mode :

```
Quo(x^3+x^2+1,2*x^2+4) mod 13
```

Output :

```
(-6)*x-6
```

Input in Maple mode :

```
Quo(x^2+2*x,x^2+6*x+5) mod 5
```

Output :

```
1
```

5.33.2 Euclidean remainder: Rem

Rem is the inert form of rem.
Rem returns the euclidean remainder between two polynomials without evaluation. It is used in conjunction with mod in Maple syntax mode to compute the euclidean remainder of the division of two polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$.
Input in Xcas mode :

```
Rem((x^3+x^2+1) mod 13,(2*x^2+4) mod 13)
```

Output :

```
rem((x^3+x^2+1)%13,(2*x^2+4)%13)
```

you need to eval(ans()) to get :

```
(-2%13)*x+-1%13
```

Input in Maple mode :

```
Rem(x^3+x^2+1,2*x^2+4) mod 13
```

Output :

```
(-2)*x-1
```

Input in `Maple` mode :

```
Rem(x^2+2*x,x^2+6*x+5) mod 5
```

Output :

```
1*x
```

5.33.3 GCD in $\mathbb{Z}/p\mathbb{Z}[x]$: `Gcd`

`Gcd` is the inert form of `gcd`.
`Gcd` returns the gcd (greatest common divisor) of two polynomials (or of a list of polynomials or of a sequence of polynomials) without evaluation.
It is used in conjunction with `mod` in Maple syntax mode to compute the gcd of two polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$ with p prime (see also 5.26.7).
Input in `Xcas` mode :

```
Gcd((2*x^2+5,5*x^2+2*x-3)%13)
```

Output :

```
gcd((2*x^2+5)%13,(5*x^2+2*x-3)%13)
```

you need to `eval(ans())` to get :

```
(1%13)*x+2%13
```

Input in `Maple` mode :

```
Gcd(2*x^2+5,5*x^2+2*x-3) mod 13
```

Output :

```
1*x+2
```

Input:

```
Gcd(x^2+2*x,x^2+6*x+5) mod 5
```

Output :

```
1*x
```

5.33.4 Factorization in $\mathbb{Z}/p\mathbb{Z}[x]$: Factor

Factor is the inert form of factor.
Factor takes as argument a polynomial.
Factor returns factor without evaluation. It is used in conjunction with mod in Maple syntax mode to factorize a polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$ where p must be prime.
Input in Xcas mode :

```
Factor((-3*x^3+5*x^2-5*x+4)%13)
```

Output :

```
factor((-3*x^3+5*x^2-5*x+4)%13)
```

you need to eval(ans()) to get :

```
((1%13)*x+-6%13)*((-3%13)*x^2+-5%13)
```

Input in Maple mode :

```
Factor(-3*x^3+5*x^2-5*x+4) mod 13
```

Output :

```
-3*(1*x-6)*(1*x^2+6)
```

5.33.5 Determinant of a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$: Det

Det is the inert form of det.
Det takes as argument a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$.
Det returns det without evaluation. It is used in conjunction with mod in Maple syntax mode to find the determinant of a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$.
Input in Xcas mode :

```
Det([[1,2,9] mod 13,[3,10,0] mod 13,[3,11,1] mod 13])
```

Output :

```
det([[1%13,2%13,-4%13],[3%13,-3%13,0%13],
        [3%13,-2%13,1%13]])
```

you need to eval(ans()) to get :

```
5%13
```

hence, in $\mathbb{Z}/13\mathbb{Z}$, the determinant of $A = [[1, 2, 9], [3, 10, 0], [3, 11, 1]]$ is 5%13 (in $\mathbb{Z}$, det(A)=31).
Input in Maple mode :

```
Det([[1,2,9],[3,10,0],[3,11,1]]) mod 13
```

Output :

5.33.6 Inverse of a matrix in $\mathbb{Z}/p\mathbb{Z}$: `Inverse`

`Inverse` is the inert form of `inverse`.
`Inverse` takes as argument a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$.
`Inverse` returns `inverse` without evaluation. It is used in conjunction with `mod` in Maple syntax mode to find the inverse of a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$.
Input in `Xcas` mode :

```
Inverse([[1,2,9] mod 13,[3,10,0] mod 13,[3,11,1]
mod13])
```

Output :

```
inverse([[1%13,2%13,9%13],[3%13,10%13,0%13],
[3%13,11%13,1%13]])
```

you need to `eval(ans())` to get :

```
[[2%13,-4%13,-5%13],[2%13,0%13,-5%13],
[-2%13,-1%13,6%13]]
```

which is the inverse of $A = [[1, 2, 9], [3, 10, 0], [3, 11, 1]]$ in $\mathbb{Z}/13\mathbb{Z}$.
Input in `Maple` mode :

```
Inverse([[1,2,9],[3,10,0],[3,11,1]]) mod 13
```

Output :

```
[[2,-4,-5],[2,0,-5],[-2,-1,6]]
```

5.33.7 Row reduction to echelon form in $\mathbb{Z}/p\mathbb{Z}$: `Rref`

`Rref` is the inert form of `rref`.
`Rref` returns `rref` without evaluation. It is used in conjunction with `mod` in Maple syntax mode to find the row reduction to echelon form of a matrix with coefficients in $\mathbb{Z}/p\mathbb{Z}$ (see also 5.54.3).
Example, solve in $\mathbb{Z}/13\mathbb{Z}$

$$\begin{cases} x + 2 \cdot y &= 9 \\ 3 \cdot x + 10 \cdot y &= 0 \end{cases}$$

Input in `Xcas` mode :

```
Rref([[1,2,9] mod 13,[3,10,0] mod 13])
```

Output :

```
rref([[1%13, 2%13, 9%13],[3%13,10%13,0%13]])
```

you need to `eval(ans())` to get :

```
[[1%13,0%13,3%13],[0%13,1%13,3%13]]
```

and conclude that `x=3%13` and `y=3%13`.
Input in `Maple` mode :

```
Rref([[1,2,9],[3,10,0],[3,11,1]]) mod 13
```

Output :

```
[[1,0,0],[0,1,0],[0,0,1]]
```

5.34 Taylor and asymptotic expansions

5.34.1 Division by increasing power order : divpc

divpc takes three arguments : two polynomials expressions A, B depending on x, such that the constant term of B is not 0, and an integer n.
divpc returns the quotient Q of the division of A by B by increasing power order, with $\text{degree}(Q) \leq n$ or $Q = 0$. In other words, Q is the Taylor expansion of order n of $\frac{A}{B}$ in the vicinity of $x = 0$.
Input :

```
divpc(1+x^2+x^3,1+x^2,5)
```

Output :

```
-x^5+x^3+1
```

Note that this command does not work on polynomials written as a list of coefficients.

5.34.2 Taylor expansion : taylor

taylor takes from one to four arguments :

- an expression depending of a variable (by default x),
- an equality variable=value (e.g. $x = a$) where to compute the Taylor expansion, by default x=0,
- an integer n, the order of the series expansion, by default 5
- a direction -1, 1 (for unidirectional series expansion) or 0 (for bidirectional series expansion) (by default 0).

Note that the syntax ...,x,n,a,... (instead of ...,x=a,n,...) is also accepted.
taylor returns a polynomial in x-a, plus a remainder of the form:
(x-a)^n*order_size(x-a)
where order_size is a function such that,

$$\forall r > 0, \quad \lim_{x \to 0} x^r \text{order\_size}(x) = 0$$

For regular series expansion, order_size is a bounded function, but for non regular series expansion, it might tend slowly to infinity, for example like a power of $\ln(x)$.
Input :

```
taylor(sin(x),x=1,2)
```

Or (be careful with the order of the arguments !) :

```
taylor(sin(x),x,2,1)
```

Output :

```
sin(1)+cos(1)*(x-1)+(-(1/2*sin(1)))*(x-1)^2+
            (x-1)^3*order_size(x-1)
```

Remark
The order returned by `taylor` may be smaller than n if cancellations between numerator and denominator occur, for example

$$\text{taylor}(\frac{x^3+\sin(x)^3}{x-\sin(x)})$$

Input :

```
taylor(x^3+sin(x)^3/(x-sin(x)))
```

The output is only a 2nd-order series expansion :

```
6+-27/10*x^2+x^3*order_size(x)
```

Indeed the numerator and denominator valuation is 3, hence we lose 3 orders. To get order 4, we should use $n = 7$.
Input :

```
taylor(x^3+sin(x)^3/(x-sin(x)),x=0,7)
```

Output is a 4th-order series expansion :

```
6+-27/10*x^2+x^3+711/1400*x^4+x^5*order_size(x)
```

5.34.3 Series expansion : `series`

`series` takes from one to four arguments :

- an expression depending of a variable (by default `x`),
- an equality variable=value (e.g. $x = a$) where to compute the series expansion, by default `x=0`,
- an integer n, the order of the series expansion, by default `5`
- a direction `-1, 1` (for unidirectional series expansion) or `0` (for bidirectional series expansion) (by default `0`).

Note that the syntax `...,x,`a`,`n`,...` (instead of `...,x=`a`,`n`,...`) is also accepted.
`series` returns a polynomial in `x-a`, plus a remainder of the form:

```
(x-a)^n*order_size(x-a)
```

where `order_size` is a function such that,

$$\forall r > 0, \quad \lim_{x\to 0} x^r \text{order\_size}(x) = 0$$

The order returned by `series` may be smaller than n if cancellations between numerator and denominator occur.
Examples :

- series expansion in the vicinity of x=0
  Find an series expansion of $\frac{x^3+\sin(x)^3}{x-\sin(x)}$ in the vicinity of x=0.
  Input :

  ```
  series(x^3+sin(x)^3/(x-sin(x)))
  ```

  Output is only a 2nd-order series expansion :

  ```
  6+-27/10*x^2+x^3*order_size(x)
  ```

  We have lost 3 orders because the valuation of the numerator and denominator is 3. To get a 4-th order expansion, we must therefore take $n = 7$.
  Input :

  ```
  series(x^3+sin(x)^3/(x-sin(x)),x=0,7)
  ```

  or :

  ```
  series(x^3+sin(x)^3/(x-sin(x)),x,0,7)
  ```

  Output is a 4th-order series expansion :

  ```
  6+-27/10*x^2+x^3+711/1400*x^4+ x^5*order_size(x)
  ```

- series expansion in the vicinity of x=a
  Find a series 4th-order expansion of $\cos(2x)^2$ in the vicinity of $x = \frac{\pi}{6}$.
  Input:

  ```
  series(cos(2*x)^2,x=pi/6, 4)
  ```

  Output :

  ```
  1/4+(-(4*sqrt(3)))/4*(x-pi/6)+(4*3-4)/4*(x-pi/6)^2+
  32*sqrt(3)/3/4*(x-pi/6)^3+(-16*3+16)/3/4*(x-pi/6)^4+
              (x-pi/6)^5*order_size(x-pi/6)
  ```

- series expansion in the vicinity of x=+∞ or x=-∞

  1. Find a 5th-order series expansion of $\arctan(x)$ in the vicinity of x=+∞.
     Input :

     ```
     series(atan(x),x=+infinity,5)
     ```

     Output :

     ```
     pi/2-1/x+1/3*(1/x)^3+1/-5*(1/x)^5+
             (1/x)^6*order_size(1/x)
     ```

     Note that the expansion variable and the argument of the `order_size` function is $h = \frac{1}{x} \rightarrow_{x\rightarrow+\infty} 0$.

2. Find a series 2nd-order expansion of $(2x-1)e^{\frac{1}{x-1}}$ in the vicinity of x=+∞.
   Input :

   ```
   series((2*x-1)*exp(1/(x-1)),x=+infinity,3)
   ```

   Output is only a 1st-order series expansion :

   ```
   2*x+1+2/x+(1/x)^2*order_size(1/x)
   ```

   To get a 2nd-order series expansion in $1/x$, input:

   ```
   series((2*x-1)*exp(1/(x-1)),x=+infinity,4)
   ```

   Output :

   ```
   2*x+1+2/x+17/6*(1/x)^2+(1/x)^3*order_size(1/x)
   ```

3. Find a 2nd-order series expansion of $(2x-1)e^{\frac{1}{x-1}}$ in the vicinity of x=-∞.
   Input :

   ```
   series((2*x-1)*exp(1/(x-1)),x=-infinity,4)
   ```

   Output :

   ```
   -2*(-x)+1-2*(-1/x)+17/6*(-1/x)^2+
       (-1/x)^3*order_size(-1/x)
   ```

- unidirectional series expansion.
  The fourth parameter indicates the direction :
  - `1` to do an series expansion in the vicinity of $x=a$ with $x>a$,
  - `-1` to do an series expansion in the vicinity of $x=a$ with $x<a$,
  - `0` to do an series expansion in the vicinity of $x=a$ with $x\neq a$.

  For example, find a 2nd-order series expansion of $\frac{(1+x)^{\frac{1}{x}}}{x^3}$ in the vicinity of $x=0^+$.
  Input :

  ```
  series((1+x)^(1/x)/x^3,x=0,2,1)
  ```

  Output :

  ```
  exp(1)/x^3+(-(exp(1)))/2/x^2+1/x*order_size(x)
  ```

5.34.4 The residue of an expression at a point : `residue`

`residue` takes as argument an expression depending on a variable, the variable name and a complex a or an expression depending on a variable and the equality : variable_name=a. `residue` returns the residue of this expression at the point a.
Input :

```
residue(cos(x)/x^3,x,0)
```

or :

```
residue(cos(x)/x^3,x=0)
```

Output :

```
(-1)/2
```

5.34.5 Padé expansion: `pade`

`pade` takes 4 arguments

- an expression,
- the variable name the expression depends on,
- an integer n or a polynomial N,
- an integer p.

`pade` returns a rational fraction P/Q such that `degree(P)`$< p$ and $P/Q = f \pmod{x^{n+1}}$ or $P/Q = f \pmod{N}$. In the first case, it means that P/Q and f have the same Taylor expansion at 0 up to order n.
Input :

```
pade(exp(x),x,5,3)
```

or :

```
pade(exp(x),x,x^6,3)
```

Output :

```
(3*x^2+24*x+60)/(-x^3+9*x^2-36*x+60)
```

To verify input :

```
taylor((3*x^2+24*x+60)/(-x^3+9*x^2-36*x+60))
```

Output :

```
1+x+1/2*x^2+1/6*x^3+1/24*x^4+1/120*x^5+x^6*order_size(x)
```

which is the 5th-order series expansion of `exp(x)` at $x = 0$.
Input :

```
pade((x^15+x+1)/(x^12+1),x,12,3)
```

or :

```
pade((x^15+x+1)/(x^12+1),x,x^13,3)
```

Output :

```
x+1
```

Input :

```
pade((x^15+x+1)/(x^12+1),x,14,4)
```

or :

```
pade((x^15+x+1)/(x^12+1),x,x^15,4)
```

Output :

```
(-2*x^3-1)/(-x^11+x^10-x^9+x^8-x^7+x^6-x^5+x^4-
x^3-x^2+x-1)
```

To verify, input :

```
series(ans(),x=0,15)
```

Output :

```
1+x-x^12-x^13+2x^15+x^16*order_size(x)
```

then input :

```
series((x^15+x+1)/(x^12+1),x=0,15)
```

Output :

```
1+x-x^12-x^13+x^15+x^16*order_size(x)
```

These two expressions have the same 14th-order series expansion at $x = 0$.

5.35 Intervals

5.35.1 Definition of an interval : `a1..a2`

An interval is represented by two real numbers separated by `..`, for example

```
1..4
1.2..sqrt(2)
```

Input :

```
A:=1..4
```

```
B:=1.2..sqrt(2)
```

Warning!
The order of the boundaries of the interval is significant. For example, if you input

```
B:=2..3; C:=3..2,
```

then `B` and `C` are different, `B==C` returns `0`.

5.35.2 Boundaries of an interval : `left right`

`left` (resp. `right`) takes as argument an interval.
`left` (resp. `right`) returns the left (resp. right) boundary of this interval.
Note that `..` is an infixed operator, therefore:

- `sommet(1..5)` is equal to `'..'` and `feuille(1..5)` is equal to `(1,5)`.
- the name of the interval followed by `[0]` returns the operator `..`
- the name of the interval followed by `[1]` (or the `left` command) returns the left boundary.
- The name of the interval followed by `[2]` (or the `right` command) returns the right boundary.

Input :

```
(3..5)[0]
```

or :

```
sommet(3..5)
```

Output :

```
'..'
```

Input :

```
left(3..5)
```

or :

```
(3..5)[1]
```

or :

```
feuille(3..5)[0]
```

or :

```
op(3..5)[0]
```

Output :

```
3
```

Input :

```
right(3..5)
```

or :

```
(2..5)[2]
```

or :

```
feuille(3..5)[1]
```

or :

```
op(3..5)[1]
```

Output :

```
5
```

Remark
`left` (resp. `right`) returns also the left (resp. right) member of an equation (for example `left(2*x+1=x+2)` returns `2*x+1`).

5.35.3 Center of an interval : `interval2center`

`interval2center` takes as argument an interval or a list of intervals.
`interval2center` returns the center of this interval or the list of centers of these intervals.
Input :

```
interval2center(3..5)
```

Output :

```
4
```

Input :

```
interval2center([2..4,4..6,6..10])
```

Output :

```
[3,5,8]
```

5.35.4 Intervals defined by their center : `center2interval`

`center2interval` takes as argument a vector `V` of reals and optionally a real as second argument (by default `V[0]-(V[1]-V[0])/2`).
`center2interval` returns a vector of intervals having the real values of the first argument as centers, where the value of the second argument is the left boundary of the first interval.
Input :

```
center2interval([3,5,8])
```

Or (since the default value is 3-(5-3)/2=2) :

```
center2interval([3,5,8],2)
```

Output :

```
[2..4,4..6,6..10]
```

Input :

```
center2interval([3,5,8],2.5)
```

Output :

```
[2.5..3.5,3.5..6.5,6.5..9.5]
```

5.36 Sequences

5.36.1 Definition : `seq[]` `()`

A sequence is represented by a sequence of elements separated by commas, without delimiters or with either `( )` or `seq[...]` as delimiters, for example

```
(1,2,3,4)
seq[1,2,3,4]
```

Input :

`A:=(1,2,3,4)` or `A:=seq[1,2,3,4]`

`B:=(5,6,3,4)` or `B:=seq[5,6,3,4]`

Remarks

- The order of the elements of the sequence is significant. For example, if `B:=(5,6,3,4)` and `C:=(3,4,5,6)`, then `B==C` returns `0`.
- (see also 5.36.5)
  `seq([0,2])=(0,0)` and `seq([0,1,1,5])=[0,0,0,0,0]` but
  `seq[0,2]=(0,2)` and `seq[0,1,1,5]=(0,1,1,5)`

5.36.2 Concat two sequences : `,`

The infix operator `,` concatenates two sequences.
Input :

```
A:=(1,2,3,4)
B:=(5,6,3,4)
A,B
```

Output :

```
(1,2,3,4,5,6,3,4)
```

5.36.3 Get an element of a sequence : `[]`

The elements of a sequence have indexes beginning at 0 in `Xcas` mode or 1 in other modes.
A sequence or a variable name assigned to a sequence followed by `[n]` returns the element of index `n` of the sequence.
Input :

```
(0,3,2)[1]
```

Output :

```
3
```

5.36.4 Sub-sequence of a sequence : `[]`

A sequence or a variable name assigned to a sequence followed by `[n1..n2]` returns the sub-sequence of this sequence starting at index `n1` and ending at index `n2`.
Input :

```
(0,1,2,3,4)[1..3]
```

Output :

```
(1,2,3)
```

5.36.5 Make a sequence or a list : `seq $`

`seq` takes two, three, four or five arguments : the first argument is an expression depending of a parameter (for example j) and the remaining argument(s) describe which values of j will be used to generate the sequence. More precisely j is assumed to move from a to b:

- with a default step of 1 or -1: `j=a..b` or `j,a..b` (Maple-like syntax), `j,a,b` (TI-like syntax)
- or with a specific step: `j=a..b,p` (Maple-like syntax), `j,a,b,p` (TI-like syntax).

If the Maple-like syntax is used, `seq` returns a sequence, if the TI-like syntax is used, `seq` returns a list.

`$` is the infixed version of `seq` when `seq` has only two arguments and always returns a sequence.
Remark:

- In `Xcas` mode, the precedence of `$` is not the same as for example in `Maple`, in case of doubt put the arguments of `$` in parenthesis. For example, the equivalent of `seq(j^2,j=-1..3)` is `(j^2)$(j=-1..3)` and returns `(1,0,1,4,9)`. The equivalent of `seq(4,3)` is `4$3` and returns `(4,4,4)`.
- With `Maple` syntax, `j,a..b,p` is not valid. To specify a step p for the variation of j from a to b, use `j=a..b,p` or use the `TI` syntax `j,a,b,p` and get the sequence from the list with `op(...)`.

In summary, the different way to build a sequence are :

- with `Maple`-like **syntax**
  1. `seq` has two arguments, either an expression depending on a parameter (for example j) and $j = a..b$ where a and b are reals, or a constant expression and an integer n.
     `seq` returns the sequence where j is replaced in the expression by a, $a + 1$,...,b if $b > a$ and by a, $a - 1$,...,b if $b < a$, or `seq` returns the sequence made by copying the constant n times.

2. `seq` has three arguments, an expression depending on a parameter (for example j) and $j = a..b, p$ where a, b are reals and p is a real number. `seq` returns the sequence where j is replaced in the expression by a, $a + p$,...,b if $b > a$ and by a, $a - p$,...,b if $b < a$.
   Note that $j, a..b$ is also valid but $j, a..b, p$ is not valid.

- `TI` **syntax**

1. `seq` has four arguments, an expression depending on a parameter (for example j), the name of the parameter (for example j), a and b where a and b are reals.
   `seq` returns the list where j is replaced in the expression by a, $a+1$,...,b if $b > a$ and by a, $a - 1$,...,b if $b < a$.
2. `seq` has five arguments, an expression depending on a parameter (for example j), the name of the parameter (for example j), a, b and p where a, b and p are reals.
   `seq` returns the list where j is substituted in the expression by a, $a + p$,...,$a + k * p$ ($a + k * p \leq b < a + (k + 1) * p$ or $a + k * p \geq b > a + (k + 1) * p$). By default, p=1 if $b > a$ and p=-1 if $b < a$.

Note that in `Maple` syntax, `seq` takes no more than 3 arguments and returns a sequence, while in `TI` syntax, `seq` takes at least 4 arguments and returns a list.
Input to have a sequence with same elements :

```
seq(t,4)
```

or :

```
seq(t,k=1..4)
```

or :

```
t$4
```

Output :

```
(t,t,t,t)
```

Input to have a sequence :

```
seq(j^3,j=1..4)
```

or :

```
(j^3)$(j=1..4)
```

or :

```
seq(j^3,j,1..4)
```

Output :

```
(1,8,27,64)
```

Input to have a sequence :

```
seq(j^3,j=-1..4,2)
```

Output :

```
(-1,1,27)
```

Or to have a list,
Input :

```
seq(j^3,j,1,4)
```

Output :

```
[1,8,27,64]
```

Input :

```
seq(j^3,j,0,5,2)
```

Output :

```
[0,8,64]
```

Input :

```
seq(j^3,j,5,0,-2)
```

or

```
seq(j^3,j,5,0,2)
```

Output :

```
[125,27,1]
```

Input :

```
seq(j^3,j,1,3,0.5)
```

Output :

```
[1,3.375,8,15.625,27]
```

Input :

```
seq(j^3,j,1,3,1/2)
```

Output :

```
[1,27/8,8,125/8,27]
```

Examples

- Find the third derivative of $\ln(t)$, input:

  ```
  diff(log(t),t$3)
  ```

  Output :

-((-(2*t))/t^4)

- Input :

  l:=[[2,3],[5,1],[7,2]]

  seq((l[k][0])$(l[k][1]),k=0 .. size(l)-1)

  Output :

  2,2,2,seq[5],7,7

  then `eval(ans())` returns:

  2,2,2,5,7,7

- Input to transform a string into the list of its characters :

```
f(chn):={
 local l;
 l:=size(chn);
 return seq(chn[j],j,0,l-1);
}
```

  then input:

  f("abracadabra")

  Output :

  ["a","b","r","a","c","a","d","a","b","r","a"]

5.36.6 Transform a sequence into a list : `[]` `nop`

To transform a sequence into list, just put square brackets (`[]`) around the sequence or use the command `nop`.
Input :

[seq(j^3,j=1..4)]

or :

seq(j^3,j,1,4)

or :

[(j^3)$(j=1..4)]

Output :

[1,4,9,16]

Input :

nop(1,4,9,16)

Output :

[1,4,9,16]

5.36.7 The + operator applied on sequences

The infixed operator +, with two sequences as argument, returns the total sum of the elements of the two sequences.
Note the difference with the lists, where the term by term sums of the elements of the two lists would be returned.
Input :

`(1,2,3,4,5,6)+(4,3,5)`

or :

`'+'((1,2,3,4,5,6),(4,3,5))`

Output :

`33`

But input :

`[1,2,3,4,5,6]+[4,3,5]`

Output :

`[5,5,8,4,5,6]`

Warning
When the operator + is prefixed, it has to be quoted (`'+'`).

5.37 Sets

5.37.1 Definition : `set[]`

To define a set of elements, put the elements separated by a comma, with `%{ ... %}` or `set[ ... ]` as delimiters.
Input :

`%{1,2,3,4%}`
`set[1,2,3,4]`

In the `Xcas` answers, the set delimiters are displayed as ⟦ and ⟧ in order not to confuse sets with lists. For example, ⟦1,2,3⟧ is the set `%{1,2,3%}`, unlike [1,2,3] (normal brackets) which is the list `[1,2,3]`.
Input :

`A:=%{1,2,3,4%}` or `A:=set[1,2,3,4]`

Output :

⟦`1,2,3,4`⟧

Input :

`B:=%{5,5,6,3,4%}` or `B:=set[5,5,6,3,4]`

Output :

⟦5,6,3,4⟧

Remark
The order in a set is not significant and the elements in a set are all distinct. If you input `B:=%{5,5,6,3,4%}` and `C:=%{3,4,5,3,6%}`, then `B==C` will return `1`.

5.37.2 Union of two sets or of two lists : `union`

`union` is an infixed operator.
`union` takes as argument two sets or two lists, `union` returns the union set of the arguments.
Input :

`set[1,2,3,4] union set[5,6,3,4]`

or :

`%{1,2,3,4%} union %{5,6,3,4%}`

Output :

⟦1,2,3,4,5,6⟧

Input :

`[1,2,3] union [2,5,6]`

Output :

⟦1,2,3,5,6⟧

5.37.3 Intersection of two sets or of two lists : `intersect`

`intersect` is an infixed operator.
`intersect` takes as argument two sets or two lists.
`intersect` returns the intersection set of the arguments.
Input :

`set[1,2,3,4] intersect set[5,6,3,4]`

or :

`%{1,2,3,4%} intersect %{5,6,3,4%}`

Output :

⟦3,4⟧

Input :

`[1,2,3,4] intersect [5,6,3,4]`

Output :

⟦3,4⟧

5.37.4 Difference of two sets or of two lists : `minus`

`minus` is an infixed operator.
`minus` takes as argument two sets or two lists.
`minus` returns the difference set of the arguments.
Input :

```
set[1,2,3,4] minus set[5,6,3,4]
```

or :

```
%{1,2,3,4%} minus %{5,6,3,4%}
```

Output :

⟦1,2⟧

Input :

```
[1,2,3,4] minus [5,6,3,4]
```

Output :

⟦1,2⟧

5.38 Lists and vectors

5.38.1 Definition

A list (or a vector) is delimited by `[ ]`, its elements must be separated by commas. For example, `[1,2,5]` is a list of three integers.

Lists can contain lists (for example, a matrix is a list of lists of the same size). Lists may be used to represent vectors (list of coordinates), matrices, univariate polynomials (list of coefficients by decreasing order).

Lists are different from sequences, because sequences are flat : an element of a sequence cannot be a sequence. Lists are different from sets, because for a list, the order is important and the same element can be repeated in a list (unlike in a set where each element is unique).

In `Xcas` output :

- vector (or list) delimiters are displayed as `[]`,
- matrix delimiters are displayed as **[]**,
- polynomial delimiters are displayed as [] [],
- set delimiters are displayed as ⟦ ⟧.

The list elements are indexed starting from 0 in Xcas syntax mode and from 1 in all other syntax modes.

5.38.2 Get an element or a sub-list of a list : `at []`

Get an element

The n-th element of a list `l` of size s is addressed by `l[n]` where n is in $[0..s-1]$ or $[1..s]$. The equivalent prefixed function is `at`, which takes as argument a list and an integer `n`.
`at` returns the element of the list at index `n`.
Input :

```
[0,1,2][1]
```

or :

```
at([0,1,2],1)
```

Output :

```
1
```

Extract a sub-list

If l is a list of size s, `l[n1..n2]` returns the list extracted from `l` containing the elements of indexes n_1 to n_2 where $0 \leq n_1 \leq n_2 < s$ (in Xcas syntax mode) or $0 < n_1 \leq n_2 \leq s$ in other syntax modes. The equivalent prefixed function is `at` with a list and an interval of integers (`n1..n2`) as arguments.
See also : `mid`, section 5.38.3.
Input :

```
[0,1,2,3,4][1..3]
```

or :

```
at([0,1,2,3,4],1..3)
```

Output :

```
[1,2,3]
```

Warning
`at` can not be used for sequences, index notation must be used, as in `(0,1,2,3,4,5)[2..3]`.

5.38.3 Extract a sub-list : `mid`

See also : `at` section 5.38.2.
`mid` is used to extract a sub-list of a list.
`mid` takes as argument a list, the index of the beginning of the sub-list and the length of the sub-list.
`mid` returns the sub-list.
Input :

```
mid([0,1,2,3,4,5],2,3)
```

Output :

```
[1,2,3]
```

Warning
`mid` can not be used to extract a subsequence of a sequence, because the arguments of `mid` would be merged with the sequence. Index notation must be used, like e.g. `(0,1,2,3,4,5)[2..3]`.

5.38.4 Get the first element of a list : `head`

`head` takes as argument a list.
`head` returns the first element of this list.
Input :

```
head([0,1,2,3])
```

Output :

```
0
```

`a:=head([0,1,2,3])` does the same thing as `a:=[0,1,2,3][0]`

5.38.5 Remove an element in a list : `suppress`

`suppress` takes as argument a list and an integer `n`.
`suppress` returns the list where the element of index `n` is removed.
Input :

```
suppress([3,4,2],1)
```

Output :

```
[3,2]
```

5.38.6 Remove the first element : `tail`

`tail` takes as argument a list. `tail` returns the list without its first element.
Input :

```
tail([0,1,2,3])
```

Output :

```
[1,2,3]
```

`l:=tail([0,1,2,3])` does the same thing as `l:=suppress([0,1,2,3],0)`

5.38.7 Reverse order in a list : `revlist`

`revlist` takes as argument a list (resp. sequence).
`revlist` returns the list (resp. sequence) in the reverse order.
Input :

```
revlist([0,1,2,3,4])
```

Output :

```
[4,3,2,1,0]
```

Input :

```
revlist([0,1,2,3,4],3)
```

Output :

```
3,[0,1,2,3,4]
```

5.38.8 Reverse a list starting from its n-th element : `rotate`

`rotate` takes as argument a list and an integer `n` (by default `n=-1`).
`rotate` rotates the list by `n` places to the left if `n>0` or to the right if `n<0`. Elements leaving the list from one side come back on the other side. By default `n=-1` and the last element becomes first.
Input :

```
rotate([0,1,2,3,4])
```

Output :

```
[4,0,1,2,3]
```

Input :

```
rotate([0,1,2,3,4],2)
```

Output :

```
[2,3,4,0,1]
```

Input :

```
rotate([0,1,2,3,4],-2)
```

Output :

```
[3,4,0,1,2]
```

5.38.9 Permuted list from its n-th element : shift

shift takes as argument a list l and an integer n (by default n=-1).
shift rotates the list to the left if n>0 or to the right if n<0. Elements leaving the list from one side are replaced by undef on the other side.
Input :

```
shift([0,1,2,3,4])
```

Output :

```
[undef,0,1,2,3]
```

Input :

```
shift([0,1,2,3,4],2)
```

Output :

```
[2,3,4,undef,undef]
```

Input :

```
shift([0,1,2,3,4],-2)
```

Output :

```
[undef,undef,0,1,2]
```

5.38.10 Modify an element in a list : subsop

subsop modifies an element in a list. subsop takes as argument a list and an equality (an index=a new value) in all syntax modes, but in Maple syntax mode the order of the arguments is reversed.
Remark If the second argument is 'k=NULL', the element of index k is removed of the list.
Input in Xcas mode (the index of the first element is 0) :

```
subsop([0,1,2],1=5)
```

or :

```
L:=[0,1,2];L[1]:=5
```

Output :

```
[0,5,2]
```

Input in Xcas mode (the index of the first element is 0) :

```
subsop([0,1,2],'1=NULL')
```

Output :

```
[0,2]
```

Input in Mupad TI mode (the index of the first element is 1) :

```
subsop([0,1,2],2=5)
```

or :

```
L:=[0,1,2];L[2]:=5
```

Output :

```
[0,5,2]
```

In `Maple` mode the arguments are permuted and the index of the first element is 1.
Input :

```
subsop(2=5,[0,1,2])
```

or :

```
L:=[0,1,2];L[2]:=5
```

Output :

```
[0,5,2]
```

5.38.11 Transform a list into a sequence : `op makesuite`

`op` or `makesuite` takes as argument a list.
`op` or `makesuite` transforms this list into a sequence.
See 5.15.3 for other usages of `op`.
Input :

```
op([0,1,2])
```

or :

```
makesuite([0,1,2])
```

Output :

```
(0,1,2)
```

5.38.12 Transform a sequence into a list : `makevector []`

Square brackets put around a sequence transform this sequence into a list or vector. The equivalent prefixed function is `makevector` which takes a sequence as argument.
`makevector` transforms this sequence into a list or vector.
Input :

```
makevector(0,1,2)
```

Output :

```
[0,1,2]
```

Input :

```
a:=(0,1,2)
```

Input :

```
[a]
```

or :

```
makevector(a)
```

Output :

```
[0,1,2]
```

5.38.13 Length of a list : `size nops length`

`size` or `nops` or `length` takes as argument a list (resp. sequence).
`size` or `nops` or `length` returns the length of this list (resp. sequence).
Input :

```
nops([3,4,2])
```

or :

```
size([3,4,2])
```

or :

```
length([3,4,2])
```

Output :

```
3
```

5.38.14 Sizes of a list of lists : `sizes`

`sizes` takes as argument a list of lists.
`sizes` returns the list of the lengths of these lists.
Input :

```
sizes([[3,4],[2]])
```

Output :

```
[2,1]
```

5.38.15 Concatenate two lists or a list and an element : `concat augment`

`concat` (or `augment`) takes as argument a list and an element or two lists.
`concat` (or `augment`) concats this list and this element, or concats these two lists.
Input :

```
concat([3,4,2],[1,2,4])
```

or :

```
augment([3,4,2],[1,2,4])
```

Output :

```
[3,4,2,1,2,4]
```

Input :

```
concat([3,4,2],5)
```

or :

```
augment([3,4,2],5)
```

Output :

```
[3,4,2,5]
```

Warning If you input :

```
concat([[3,4,2]],[[1,2,4]])
```

or

```
augment([[3,4,2]],[[1,2,4]])
```

the output will be:

```
[[3,4,2,1,2,4]]
```

5.38.16 Append an element at the end of a list : `append`

`append` takes as argument a list and an element.
`append` puts this element at the end of this list.
Input :

```
append([3,4,2],1)
```

Output :

```
[3,4,2,1]
```

Input :

```
append([1,2],[3,4])
```

Output :

```
[1,2,[3,4]]
```

5.38.17 Prepend an element at the beginning of a list : prepend

`prepend` takes as argument a list and an element.
`prepend` puts this element at the beginning of this list.
Input :

```
prepend([3,4,2],1)
```

Output :

```
[1,3,4,2]
```

Input :

```
prepend([1,2],[3,4])
```

Output :

```
[[3,4],1,2]
```

5.38.18 Sort : sort

`sort` takes as argument a list or an expression.

- For a list,
  `sort` returns the list sorted in increasing order.
  Input :

  ```
  sort([3,4,2])
  ```

  Output :

  ```
  [2,3,4]
  ```

- For an expression,
  `sort` sorts and collects terms in sums and products.
  Input :

  ```
  sort(exp(2*ln(x))+x*y-x+y*x+2*x)
  ```

  Output :

  ```
  2*x*y+exp(2*ln(x))+x
  ```

  Input :

  ```
  simplify(exp(2*ln(x))+x*y-x+y*x+2*x)
  ```

  Output :

  ```
  x^2+2*x*y+x
  ```

`sort` accepts an optional second argument, which is a bivariate function returning 0 or 1. If provided, this function will be used to sort the list, for example `(x,y)->x>=y` may be used as second argument to sort the list in decreasing order. This may also be used to sort list of lists (that `sort` with one argument does not know how to sort).
Input :

```
sort([3,4,2],(x,y)->x>=y)
```

Output :

```
[4,3,2]
```

5.38.19 Sort a list by increasing order : `SortA`

`SortA` takes as argument a list.
`SortA` returns this list sorted by increasing order.
Input :

```
SortA([3,4,2])
```

Output :

```
[2,3,4]
```

`SortA` may have a matrix as argument and in this case, `SortA` modifies the order of columns by sorting the first matrix row by increasing order.
Input :

```
SortA([[3,4,2],[6,4,5]])
```

Output :

```
[[2,3,4],[5,6,4]]
```

5.38.20 Sort a list by decreasing order : `SortD`

`SortD` takes a list as argument.
`SortD` returns this list sorted by decreasing order.
Input :

```
SortD([3,4,2])
```

Output :

```
[2,3,4]
```

`SortD` may have a matrix as argument and in this case, `SortD` modifies the order of columns by sorting the first matrix row by decreasing order.
Input :

```
SortD([[3,4,2],[6,4,5]])
```

Output :

```
[[4,3,2],[4,6,5]]
```

5.38.21 Select the elements of a list : `select`

`select` takes as arguments : a boolean function `f` and a list `L`.
`select` selects in the list `L`, the elements `c` such that `f(c)==true`.
Input :

```
select(x->(x>=2),[0,1,2,3,1,5])
```

Output :

```
[2,3,5]
```

5.38.22 Remove elements of a list : `remove`

`remove` takes as argument : a boolean function `f` and a list `L`.
`remove` removes in the list `L`, the elements `c` such that `f(c)==true`.
Input :

```
remove(x->(x>=2),[0,1,2,3,1,5])
```

Output :

```
[0,1,1]
```

Remark The same applies on strings, for example, to remove all the "a" of a string:
Input :

```
ord("a")
```

Output :

```
97
```

Input :

```
f(chn):={
  local l:=length(chn)-1;
  return remove(x->(ord(x)==97),seq(chn[k],k,0,l));
}
```

Then, input :

```
f("abracadabra")
```

Output :

```
["b","r","c","d","b","r"]
```

To get a string, input :

```
char(ord(["b","r","c","d","b","r"])
```

Output :

```
"brcdbr"
```

5.38.23 Test if a value is in a list : `member`

`member` takes as argument a value `c` and a list (or a set) `L`.
`member` is a function that tests if `c` is an element of the list `L`.
`member` returns `0` if `c` is not in `L`, or a strictly positive integer which is 1 plus the index of the first occurrence of `c` in `L`.
Note the order of the arguments (required for compatibility reasons)
Input :

```
member(2,[0,1,2,3,4,2])
```

Output :

```
3
```

Input :

```
member(2,%{0,1,2,3,4,2%})
```

Output :

```
3
```

5.38.24 Test if a value is in a list : `contains`

`contains` takes as argument a list (or a set) `L` and a value `c`.
`contains` tests if `c` is an element of the list `L`.
`contains` returns `0` if `c` is not in `L`, or a strictly positive integer which is 1+the index of the first occurrence of `c` in `L`.
Input :

```
contains([0,1,2,3,4,2],2)
```

Output :

```
3
```

Input :

```
contains(%{0,1,2,3,4,2%},2)
```

Output :

```
3
```

5.38.25 Sum of list (or matrix) elements transformed by a function : `count`

`count` takes as argument : a real function `f` and a list `l` of length `n` (or a matrix `A` of dimension `p*q`).
`count` applies the function to the list (or matrix) elements and returns their sum, i.e. :
`count(f,l)` returns `f(l[0])+f(l[1])+...+f(l[n-1])` or
`count(f,A)` returns `f(A[0,0])+....+f(A[p-1,q-1])`.
If `f` is a boolean function `count` returns the number of elements of the list (or of the matrix) for which the boolean function is true.
Input :

```
count((x)->x,[2,12,45,3,7,78])
```

Output :

```
147
```

because : 2+12+45+3+7+78=147.
Input :

```
count((x)->x<12,[2,12,45,3,7,78])
```

Output :

```
3
```

Input :

```
count((x)->x==12,[2,12,45,3,7,78])
```

Output :

```
1
```

Input :

```
count((x)->x>12,[2,12,45,3,7,78])
```

Output :

```
2
```

Input :

```
count(x->x^2,[3,5,1])
```

Output :

```
35
```

Indeed $3^2 + 5^2 + 1^1 = 35$.
Input :

```
count(id,[3,5,1])
```

Output :

```
9
```

Indeed, `id` is the identity functions and 3+5+1=9.
Input :

```
count(1,[3,5,1])
```

Output :

```
3
```

Indeed, `1` is the constant function equal to 1 and 1+1+1=3.

5.38.26 Number of elements equal to a given value : count_eq

count_eq takes as argument : a real and a real list (or matrix).
count_eq returns the number of elements of the list (or matrix) which are equal to the first argument.
Input :

```
count_eq(12,[2,12,45,3,7,78])
```

Output :

```
1
```

5.38.27 Number of elements smaller than a given value : count_inf

count_inf takes as argument : a real and a real list (or matrix).
count_inf returns the number of elements of the list (or matrix) which are strictly less than the first argument.
Input :

```
count_inf(12,[2,12,45,3,7,78])
```

Output :

```
3
```

5.38.28 Number of elements greater than a given value : count_sup

count_sup takes as argument : a real and a real list (or matrix).
count_sup returns the number of elements of the list (or matrix) which are strictly greater than the first argument.
Input :

```
count_sup(12,[2,12,45,3,7,78])
```

Output :

```
2
```

5.38.29 Sum of elements of a list : sum add

sum or add takes as argument a list l (resp. sequence) of reals.
sum or add returns the sum of the elements of l.
Input :

```
sum(2,3,4,5,6)
```

Output :

```
20
```

5.38.30 Cumulated sum of the elements of a list : cumSum

cumSum takes as argument a list l (resp. sequence) of numbers or of strings. cumSum returns the list (resp. sequence) with same length as l and with k-th element the sum (or concatenation) of the elements l[0], .., l[k].
Input :

```
cumSum(sqrt(2),3,4,5,6)
```

Output :

```
sqrt(2),3+sqrt(2),3+sqrt(2)+4,3+sqrt(2)+4+5,
3+sqrt(2)+4+5+6
```

Input :

```
normal(cumSum(sqrt(2),3,4,5,6))
```

Output :

```
sqrt(2),sqrt(2)+3,sqrt(2)+7,sqrt(2)+12,sqrt(2)+18
```

Input :

```
cumSum(1.2,3,4.5,6)
```

Output :

```
1.2,4.2,8.7,14.7
```

Input :

```
cumSum([0,1,2,3,4])
```

Output :

```
[0,1,3,6,10]
```

Input :

```
cumSum("a","b","c","d")
```

Output :

```
"a","ab","abc","abcd"
```

Input :

```
cumSum("a","ab","abc","abcd")
```

Output :

```
"a","aab","aababc","aababcabcd"
```

5.38.31 Product : product mul

See also 5.38.31, 5.43.6 and 5.43.8).

Product of values of an expression : `product`

`product(expr,var,a,b,p)` or `mul(expr,var,a,b,p)` returns the product of values of an expression `ex` when the variable `var` goes from `a` to `b` with a step `p` (by default p=1) : this syntax is for compatibility with Maple.
Input :

`product(x^2+1,x,1,4)`

or:

`mul(x^2+1,x,1,4)`

Output :

`1700`

Indeed $2 * 5 * 10 * 17 = 1700$
Input :

`product(x^2+1,x,1,5,2)`

or:

`mul(x^2+1,x,1,5,2)`

Output :

`520`

Indeed $2 * 10 * 26 = 520$

Product of elements of a list : `product`

`product` or `mul` takes as argument a list `l` of reals (or floating numbers) or two lists of the same size (see also 5.38.31, 5.43.6 and 5.43.8).

- if `product` or `mul` has a list `l` as argument, `product` or `mul` returns the product of the elements of `l`.
  Input :

  `product([2,3,4])`

  or :

  `mul([2,3,4])`

  Output :

  `24`

  Input :

  `product([[2,3,4],[5,6,7]])`

Output :

`[10,18,28]`

- if `product` or `mul` takes as arguments `l1` and `l2` (two lists or two matrices), `product` or `mul` returns the term by term product of the elements of `l1` and `l2`.
  Input :

  `product([2,3,4],[5,6,7])`

  or :

  `mul([2,3,4],[5,6,7])`

  Output :

  `[10,18,28]`

  Input :

  `product([[2,3,4],[5,6,7]],[[2,3,4],[5,6,7]])`

  or :

  `mul([[2,3,4],[5,6,7]],[[2,3,4],[5,6,7]])`

  Output :

  `[[4,9,16],[25,36,49]]`

5.38.32 Apply a function of one variable to the elements of a list : `map` `apply` `of`

`map` or `apply` or `of` applies a function to a list of elements.
`of` is the prefixed function equivalent to the parenthesis : `Xcas` translates `f(x)` internally to `of(f,x)`. It is more natural to call `map` or `apply` than `of`. Be careful with the order of arguments (that is required for compatibility reasons).
Note that `apply` returns a list (`[]`) even if the second argument is not a list.
Input :

`apply(x->x^2,[3,5,1])`

or :

`of(x->x^2,[3,5,1])`

or :

`map([3,5,1],x->x^2)`

or first define the function $h(x) = x^2$, input :

```
h(x):=x^2
```

then :

```
apply(h,[3,5,1])
```

or :

```
of(h,[3,5,1])
```

or :

```
map([3,5,1],h)
```

Output :

```
[9,25,1]
```

Next example, define the function $g(x) = [x, x^2, x^3]$, input :

```
g:=(x)->[x,x^2,x^3]
```

then :

```
apply(g,[3,5,1])
```

or :

```
of(g,[3,5,1])
```

or :

```
map([3,5,1],g)
```

Output :

```
[[3,9,27],[5,25,125],[1,1,1]]
```

Warning!!! first purge `x` if `x` is not symbolic.
Note that if `l1,l2,l3` are lists `sizes([l1,l2,l3])` is equivalent to `map(size,[l1,l2,l3])`.

5.38.33 Apply a bivariate function to the elements of two lists : `zip`

`zip` applies a bivariate function to the elements of 2 lists.
Input :

```
zip('sum',[a,b,c,d],[1,2,3,4])
```

Output :

```
[a+1,b+2,c+3,d+4]
```

Input :

```
zip((x,y)->x^2+y^2,[4,2,1],[3,5,1])
```

or :

```
f:=(x,y)->x^2+y^2
```

then,

```
zip(f,[4,2,1],[3,5,1])
```

Output :

```
[25,29,2]
```

Input :

```
f:=(x,y)->[x^2+y^2,x+y]
```

then :

```
zip(f,[4,2,1],[3,5,1])
```

Output :

```
[[25,7],[29,7],[2,2]]
```

5.38.34 Make a list with zeros : `newList`

`newList(n)` makes a list of `n` zeros.
Input :

```
newList(3)
```

Output :

```
[0,0,0]
```

5.38.35 Make a list with a function : `makelist`

`makelist` takes as argument a function `f`, the bounds `a,b` of an index variable and a step `p` (by default 1 or -1 depending on the bounds order).
`makelist` makes the list `[f(a),f(a+p)...f(a+k*p)]` with k such that : $a < a + k * p \leq b < a + (k+1) * p$ or $a > a + k * p \geq b > a + (k+1) * p$.
Input :

```
makelist(x->x^2,3,5)
```

or

```
makelist(x->x^2,3,5,1)
```

or first define the function $h(x) = x^2$ by `h(x):=x^2` then input

```
makelist(h,3,5,1)
```

Output :

```
[9,16,25]
```

Input :

```
makelist(x->x^2,3,6,2)
```

Output :

```
[9,25]
```

Warning!!! purge `x` if `x` is not symbolic.

5.38.36 Make a random vector or list : `randvector`

`randvector` takes as argument an integer n and optionally a second argument, either an integer k or the quoted name of a random distribution law (see also 5.25.25, 5.38.36 and ??).
`randvector` returns a vector of size n containing random integers uniformly distributed between -99 and +99 (default), or between 0 and $k-1$ or containing random integers according to the law put between quotes.
Input :

```
randvector(3)
```

Output :

```
[-54,78,-29]
```

Input :

```
randvector(3,5)
```

or :

```
randvector(3,'rand(5)')
```

Output :

```
[1,2,4]
```

Input :

```
randvector(3,'randnorm(0,1)')
```

Output :

```
[1.39091705476,-0.136794772167,0.187312440336]
```

Input :

```
randvector(3,2..4)
```

Output :

```
[3.92450003885,3.50059241243,2.7322040787]
```

5.38.37 List of differences of consecutive terms : `deltalist`

`deltalist` takes as argument a list.
`deltalist` returns the list of the difference of all pairs of consecutive terms of this list.
Input :

```
deltalist([5,8,1,9])
```

Output :

```
[3,-7,8]
```

5.38.38 Make a matrix with a list : list2mat

list2mat takes as argument a list l and an integer p.
list2mat returns a matrix having p columns by cutting the list l in rows of length p. The matrix is filled with 0s if the size of l is not a multiple of p.
Input :

```
list2mat([5,8,1,9,5,6],2)
```

Output :

```
[[5,8],[1,9],[5,6]]
```

Input :

```
list2mat([5,8,1,9],3)
```

Output :

```
[[5,8,1],[9,0,0]]
```

Remark
Xcas displays matrix with **[** and **]** and lists with [and] as delimiters (the vertical bar of the brackets are thicker for matrices).

5.38.39 Make a list with a matrix : mat2list

mat2list takes as argument a matrix.
mat2list returns the list of the coefficients of this matrix.
Input :

```
mat2list([[5,8],[1,9]])
```

Output :

```
[5,8,1,9]
```

5.39 Functions for vectors

5.39.1 Norms of a vector : maxnorm l1norm l2norm norm

The instructions to compute the different norm of a vector are :

- maxnorm returns the l^∞ norm of a vector, defined as the maximum of the absolute values of its coordinates.
  Input :

  ```
  maxnorm([3,-4,2])
  ```

  Output :

  ```
  4
  ```

  Indeed : x=3, y=-4, z=2 and 4=max(|x|,|y|,|z|).

- `l1norm` returns the l^1 norm of a vector defined as the sum of the absolute values of its coordinates.
  Input :

  `l1norm([3,-4,2])`

  Output :

  `9`

  Indeed : `x=3, y=-4, z=2` and `9=|x|+|y|+|z|`.

- `norm` or `l2norm` returns the l^2 norm of a vector defined as the square root of the sum of the squares of its coordinates.
  Input :

  `norm([3,-4,2])`

  Output :

  `sqrt(29)`

  Indeed : `x=3, y=-4, z=2` and $29 = |x|^2 + |y|^2 + |z|^2$.

5.39.2 Normalize a vector : `normalize unitV`

`normalize` or `unitV` takes as argument a vector.
`normalize` or `unitV` normalizes this vector for the l^2 norm (the square root of the sum of the squares of its coordinates).
Input :

`normalize([3,4,5])`

Output :

`[3/(5*sqrt(2)),4/(5*sqrt(2)),5/(5*sqrt(2))]`

Indeed : `x=3, y=4, z=5` and $50 = |x|^2 + |y|^2 + |z|^2$.

5.39.3 Term by term sum of two lists : `+ .+`

The infixed operator + or `.+` and the prefixed operator `'+'` returns the term by term sum of two lists.
If the two lists do not have the same size, the smaller list is completed with zeros.
Note the difference with sequences : if the infixed operator + or the prefixed operator `'+'` takes as arguments two sequences, it merges the sequences, hence return the sum of all the terms of the two sequences.
Input :

`[1,2,3]+[4,3,5]`

or :

```
[1,2,3] .+[4,3,5]
```

or :

```
'+'([1,2,3],[4,3,5])
```

or :

```
'+'([[1,2,3],[4,3,5]])
```

Output :

```
[5,5,8]
```

Input :

```
[1,2,3,4,5,6]+[4,3,5]
```

or :

```
'+'([1,2,3,4,5,6],[4,3,5])
```

or :

```
'+'([[1,2,3,4,5,6],[4,3,5]])
```

Output :

```
[5,5,8,4,5,6]
```

Warning !
When the operator + is prefixed, it should be quoted (`'+'`).

5.39.4 Term by term difference of two lists : `-` `.-`

The infixed operator `-` or `.-` and the prefixed operator `'-'` returns the term by term difference of two lists.
If the two lists do not have the same size, the smaller list is completed with zeros.
Input :

```
[1,2,3]-[4,3,5]
```

or :

```
[1,2,3] .+ [4,3,5]
```

or :

```
'-'([1,2,3],[4,3,5])
```

or :

```
'-'([[1,2,3],[4,3,5]])
```

Output :

```
[-3,-1,-2]
```

Warning !
When the operator − is prefixed, it should be quoted (`'-'`).

5.39.5 Term by term product of two lists : .*

The infixed operator .* returns the term by term product of two lists of the same size.
Input :

```
[1,2,3] .* [4,3,5]
```

Output :

```
[4,6,15]
```

5.39.6 Term by term quotient of two lists : ./

The infixed operator ./ returns the term by term quotient of two lists of the same size.
Input :

```
[1,2,3] ./ [4,3,5]
```

Output :

```
[1/4,2/3,3/5]
```

5.39.7 Scalar product : scalar_product * dotprod dot dotP scalar_Product

dot or dotP or dotprod or scalar_product or scalarProduct or the infixed operator * takes as argument two vectors.
dot or dotP or dotprod or scalar_product or scalarProduct or * returns the scalar product of these two vectors.
Input :

```
dot([1,2,3],[4,3,5])
```

or :

```
scalar_product([1,2,3],[4,3,5])
```

or :

```
[1,2,3]*[4,3,5]
```

or :

```
'*'([1,2,3],[4,3,5])
```

Output :

```
25
```

Indeed 25=1*4+2*3+3*5.

Note that * may be used to find the product of two polynomials represented as list of their coefficients, but to avoid ambiguity, the polynomial lists must be poly1[...].

5.39.8 Cross product : `cross crossP crossproduct`

`cross` or `crossP` or `crossproduct` takes as argument two vectors.
`cross` or `crossP` or `crossproduct` returns the cross product of these two vectors.
Input :

```
cross([1,2,3],[4,3,2])
```

Output :

```
[-5,10,-5]
```

Indeed : $-5 = 2*2-3*3$, $10 = -1*2+4*3$, $-5 = 1*3-2*4$.

5.40 Statistics functions : `mean,variance,stddev, stddevp,median,quantile,quartiles,boxwhisker`

The functions described here may be used if the statistics series is contained in a list. See also section 5.43.31 for matrices and chapter **??** for weighted lists.

- `mean` computes the arithmetic mean of a list
  Input :

  ```
  mean([3,4,2])
  ```

  Output :

  ```
  3
  ```

  Input :

  ```
  mean([1,0,1])
  ```

  Output

  ```
  2/3
  ```

- `stddev` computes the standard deviation of a population, if the argument is the population.
  Input :

  ```
  stddev([3,4,2])
  ```

  Output :

  ```
  sqrt(2/3)
  ```

- `stddevp` computes an unbiased estimate of the standard deviation of the population, if the argument is a sample. The following relation holds:

`stddevp(l)^2=size(l)*stddev(l)^2/(size(l)-1).`

Input :

`stddevp([3,4,2])`

Output :

`1`

- `variance` computes the variance of a list, that is the square of `stddevp`
  Input :

  `variance([3,4,2])`

  Output :

  `2/3`

- `median` computes the median of a list.
  Input :

  `median([0,1,3,4,2,5,6])`

  Output :

  `3.0`

- `quantile` computes the deciles of a list given as first argument, where the decile is the second argument.
  Input :

  `quantile([0,1,3,4,2,5,6],0.25)`

  Output the first quartile :

  `[1.0]`

  Input :

  `quantile([0,1,3,4,2,5,6],0.5)`

  Output the median :

  `[3.0]`

  Input :

  `quantile([0,1,3,4,2,5,6],0.75)`

Output the third quartile :

`[4.0]`

- `quartiles` computes the minimum, the first quartile, the median, the third quartile and the maximum of a list.
  Input :

  `quartiles([0,1,3,4,2,5,6])`

  Output :

  `[[0.0],[1.0],[3.0],[4.0],[6.0]]`

- `boxwhisker` draws the whisker box of a statistics series stored in a list.
  Input :

  `boxwhisker([0,1,3,4,2,5,6])`

  Output

  `the graph of the whisker box of this statistic list`

Example
Define the list `A` by:

`A:=[0,1,2,3,4,5,6,7,8,9,10,11]`

Outputs :

1. `11/2` for `mean(A)`
2. `sqrt(143/12)` for `stddev(A)`
3. `0` for `min(A)`
4. `[1.0]` for `quantile(A,0.1)`
5. `[2.0]` for `quantile(A,0.25)`
6. `[5.0]` for `median(A)` or for `quantile(A,0.5)`
7. `[8.0]` for `quantile(A,0.75)`
8. `[9.0]` for `quantile(A,0.9)`
9. `11` for `max(A)`
10. `[[0.0],[2.0],[5.0],[8.0],[11.0]]` for `quartiles(A)`

5.41 Table with strings as indexes : `table`

A table is an associative container (or map), it is used to store information associated to indexes which are much more general than integers, like strings or sequences. It may be used for example to store a table of phone numbers indexed by names.
In `Xcas`, the indexes in a table may be any kind of `Xcas` objects. Access is done by a binary search algorithm, where the sorting function first sorts by `type` then uses an order for each type (e.g. $<$ for numeric types, lexicographic order for strings, etc.)
`table` takes as argument a list or a sequence of equalities `index_name=element_value`. `table` returns this table.
Input :

```
T:=table(3=-10,"a"=10,"b"=20,"c"=30,"d"=40)
```

Input :

```
T["b"]
```

Output :

```
20
```

Input :

```
T[3]
```

Output :

```
-10
```

Remark
If you assign `T[n]:= ...` where `T` is a variable name and `n` an integer

- if the variable name was assigned to a list or a sequence, then the n-th element of `T` is modified,
- if the variable name was not assigned, a table `T` is created with one entry (corresponding to the index n). Note that after the assignation `T` is not a list, despite the fact that n was an integer.

5.42 Usual matrix

A matrix is represented by a list of lists, all having the same size. In the `Xcas` answers, the matrix delimiters are **[]** (bold brackets). For example, **[**1,2,3**]** is the matrix [[1,2,3]] with only one row, unlike [1,2,3] (normal brackets) which is the list [1,2,3].
In this document, the input notation ([[1,2,3]]) will be used for input and output.

5.42.1 Identity matrix : `idn identity`

`idn` takes as argument an integer n or a square matrix.
`idn` returns the identity matrix of size n or of the same size as the matrix argument.
Input :

```
idn(2)
```

Output :

```
[[1,0],[0,1]]
```

Input :

```
idn(3)
```

Output :

```
[[1,0,0],[0,1,0],[0,0,1]]
```

5.42.2 Zero matrix : `newMat matrix`

`newMat(n,p)` or `matrix(n,p)` takes as argument two integers.
`newMat(n,p)` returns the zero matrix with `n` rows and `p` columns.
Input :

```
newMat(4,3)
```

Output :

```
[[0,0,0],[0,0,0],[0,0,0],[0,0,0]]
```

5.42.3 Random matrix : `ranm randMat randmatrix`

`ranm` or `randMat` or `randmatrix` takes as argument an integer n or two integers n, m and optionally a third argument, either an integer k or the quoted name of a random distribution law (see also 5.25.25, 5.38.36 and **??**).
`ranm` returns a vector of size n or a matrix of size $n \times m$ containing random integers uniformly distributed between -99 and +99 (default), or between 0 and $k-1$ or a matrix of size $n \times m$ containing random integers according to the law put between quotes.
Input :

```
ranm(3)
```

Output :

```
[-54,78,-29]
```

Input :

```
ranm(2,4)
```

Output :

```
[[27,-29,37,-66],[-11,76,65,-33]]
```

Input :

```
ranm(2,4,3)
```

or :

```
ranm(2,4,'rand(3)')
```

Output :

```
[[0,1,1,0],[0,1,2,0]]
```

Input :

```
ranm(2,4,'randnorm(0,1)')
```

Output :

```
[[1.83785427742,0.793007112053,-0.978388964902,-1.88602023857],
[-1.50900874199,-0.241173369698,0.311373795585,-0.532752431454]]
```

Input :

```
ranm(2,4,2..4)
```

Output :

```
[[2.00549363438,3.03381264955,2.06539073586,2.04844321217],
[3.88383254968,3.28664474655,3.76909781061,2.39113253355]]
```

5.42.4 Diagonal of a matrix or matrix of a diagonal : `BlockDiagonal` `diag`

`diag` or `BlockDiagonal` takes as argument a matrix A or a list l.
`diag` returns the diagonal of A or the diagonal matrix with the list l on the diagonal (and 0 elsewhere).
Input :

```
diag([[1,2],[3,4]])
```

Output :

```
[1,4]
```

Input :

```
diag([1,4])
```

Output :

```
[[1,0],[0,4]]
```

5.42.5 Jordan block : `JordanBlock`

`JordanBlock` takes as argument an expression a and an integer n.
`JordanBlock` returns a square matrix of size n with a on the principal diagonal, 1 above this diagonal and 0 elsewhere.
Input :

```
JordanBlock(7,3)
```

Output :

```
[[7,1,0],[0,7,1],[0,0,7]]
```

5.42.6 Hilbert matrix : `hilbert`

`hilbert` takes as argument an integer n.
`hilbert` returns the Hilbert matrix.
A Hilbert matrix is a square matrix of size n whose elements $a_{j,k}$ are :

$$a_{j,k} = \frac{1}{j+k+1}, \quad 0 \leq j, 0 \leq k$$

Input :

```
hilbert(4)
```

Output :

```
[[1,1/2,1/3,1/4],[1/2,1/3,1/4,1/5],[1/3,1/4,1/5,1/6],
[1/4,1/5,1/6,1/7]]
```

5.42.7 Vandermonde matrix : `vandermonde`

`vandermonde` takes as argument a vector whose components are denoted by x_j for $j = 0..n-1$.
`vandermonde` returns the corresponding Vandermonde matrix (the k-th row of the matrix is the vector whose components are x_i^k for $i = 0..n-1$ and $k = 0..n-1$).
Warning !
The indices of the rows and columns begin at 0 with `Xcas`.
Input :

```
vandermonde([a,2,3])
```

Output (if `a` is symbolic else purge(a)) :

```
[[1,1,1],[a,2,3],[a*a,4,9]]
```

5.43 Arithmetic and matrix

5.43.1 Evaluate a matrix : `evalm`

`evalm` is used in `Maple` to evaluate a matrix. In `Xcas`, matrices are evaluated by default, the command `evalm` is only available for compatibility, it is equivalent to `eval`.

5.43.2 Addition and subtraction of two matrices : `+ - .+ .-`

The infixed operator + or `.+` (resp. − or `.-`) are used for the addition (resp. subtraction) of two matrices.
Input :

```
[[1,2],[3,4]] + [[5,6],[7,8]]
```

Output :

```
[[6,8],[10,12]]
```

Input :

```
[[1,2],[3,4]] - [[5,6],[7,8]]
```

Output :

```
[[-4,-4],[-4,-4]]
```

Remark
+ can be used as a prefixed operator, in that case + must be quoted (`'+'`).
Input :

```
'+'([[1,2],[3,4]],[[5,6],[7,8]],[[2,2],[3,3]])
```

Output :

```
[[8,10],[13,15]]
```

5.43.3 Multiplication of two matrices : `* &*`

The infixed operator `*` (or `&*`) is used for the multiplication of two matrices.
Input :

```
[[1,2],[3,4]] * [[5,6],[7,8]]
```

or :

```
[[1,2],[3,4]] &* [[5,6],[7,8]]
```

Output :

```
[[19,22],[43,50]]
```

5.43.4 Addition of elements of a column of a matrix : `sum`

`sum` takes as argument a matrix A.
`sum` returns the list whose elements are the sum of the elements of each column of the matrix A.
Input :

```
sum([[1,2],[3,4]])
```

Output :

```
[4,6]
```

5.43.5 Cumulated sum of elements of each column of a matrix : `cumSum`

`cumSum` takes as argument a matrix A.
`cumSum` returns the matrix whose columns are the cumulated sum of the elements of the corresponding column of the matrix A.
Input :

```
cumSum([[1,2],[3,4],[5,6]])
```

Output :

```
[[1,2],[4,6],[9,12]]
```

since the cumulated sums are : 1, 1+3=4, 1+3+5=9 and 2, 2+4=6, 2+4+6=12.

5.43.6 Multiplication of elements of each column of a matrix : `product`

`product` takes as argument a matrix A.
`product` returns the list whose elements are the product of the elements of each column of the matrix A (see also 5.38.31 and 5.43.8).
Input :

```
product([[1,2],[3,4]])
```

Output :

```
[3,8]
```

5.43.7 Power of a matrix : ^ &^

The infixed operator `^` (or `&^`) is used to raise a matrix to an integral power.
Input :

```
[[1,2],[3,4]] ^ 5
```

or :

```
[[1,2],[3,4]] &^ 5
```

Output :

```
[[1069,1558],[2337,3406]]
```

5.43.8 Hadamard product : `hadamard product`

`hadamard` (or `product`) takes as arguments two matrices A and B of the same size.
`hadamard` (or `product`) returns the matrix where each term is the term by term product of A and B.
Input :

```
hadamard([[1, 2],[3,4]],[[5, 6],[7, 8]])
```

Output :

```
[[5,12],[21,32]]
```

See also 5.38.31 and 5.43.6 for `product`.

5.43.9 Hadamard product (infixed version): .*

.* takes as arguments two matrices or two lists A and B of the same size.
.* is an infixed operator that returns the matrix or the list where each term is the term by term product of the corresponding terms of A and B.
Input :

```
[[1, 2],[3,4]] .* [[5, 6],[7, 8]]
```

Output :

```
[[5,12],[21,32]]
```

Input :

```
[1,2,3,4] .* [5,6,7,8]
```

Output :

```
[5,12,21,32]
```

5.43.10 Hadamard division (infixed version): ./

./ takes as arguments two matrices or two lists A and B of the same size.
./ is an infixed operator that returns the matrix or the list where each term is the term by term division of the corresponding terms of A and B.
Input :

```
[[1, 2],[3,4]] ./ [[5, 6],[7, 8]]
```

Output :

```
[[1/5,1/3],[3/7,1/2]]
```

5.43.11 Hadamard power (infixed version): .^

.^ takes as arguments a matrix or a list A and a real b.
.^ is an infixed operator that returns the matrix or the list where each term is the corresponding term of A raised to the power b.
Input :

```
[[1, 2],[3,4]] .^ 2
```

Output :

```
[[1,4],[9,16]]
```

5.43.12 Extracting element(s) of a matrix : `[]` `at`

Recall that a matrix is a list of lists with the same size.
Input :

```
A:=[[3,4,5],[1,2,6]]
```

Output :

```
[[3,4,5],[1,2,6]]
```

The prefixed function `at` or the index notation `[...]` is used to access to an element or a row or a column of a matrix:

- To extract an element, put the matrix and then, between square brackets put its row index, a comma, and its column index. In `Xcas` mode the first index is 0, in other modes the first index is 1.
  Input :

  ```
  [[3,4,5],[1,2,6]][0,1]
  ```

  or :

  ```
  A[0,1]
  ```

  or :

  ```
  A[0][1]
  ```

  or :

  ```
  at(A,[0,1])
  ```

  Output :

  ```
  4
  ```

- To extract a row of the matrix `A`, put the matrix and then, between square brackets put the row index, input :

  ```
  [[3,4,5],[1,2,6]][0]
  ```

  or :

  ```
  A[0]
  ```

  or :

  ```
  at(A,0)
  ```

  Output :

```
[3,4,2]
```

- To extract a part of a row, put two arguments between the square brackets : the row index and an interval to designate the selected columns.
  Input :

  ```
  A[1,0..2]
  ```

  Output :

  ```
  [1,2,6]
  ```

  Input :

  ```
  A[1,1..2]
  ```

  Output :

  ```
  [2,6]
  ```

- To extract a column of the matrix `A`, first transpose `A` (`transpose(A)`) then extract the row like above.
  Input :

  ```
  tran(A)[1]
  ```

  or :

  ```
  at(tran(A),1)
  ```

  Output :

  ```
  [4,2]
  ```

- To extract a part of a column of the matrix `A` as a list, put two arguments between the square brackets : an index interval to designate the selected rows and the column index.
  Input :

  ```
  A[0..0,1]
  ```

  Output :

  ```
  [4]
  ```

  This may be used to extract a full column, by specifying all the rows as an index interval.
  Input :

```
A[0..1,1]
```

Output :

```
[4,2]
```

- To extract a sub-matrix of a matrix, put between the square brackets two intervals : one interval for the selected rows and one interval for the selected columns.
  To define the matrix `A`, input :

  ```
  A:=[[3,4,5],[1,2,6]]
  ```

  Input :

  ```
  A[0..1,1..2]
  ```

  Output :

  ```
  [[4,5],[2,6]]
  ```

  Input :

  ```
  A[0..1,1..1]
  ```

  Output :

  ```
  [[4],[2]]
  ```

  Remark If the second interval is omitted, the sub-matrix is made with the consecutive rows given by the first interval.
  Input :

  ```
  A[1..1]
  ```

  Output :

  ```
  [[1,2,6]]
  ```

You may also assign an element of a matrix using index notation, if you assign with := a new copy of the matrix is created and the element is modified, if you assign with =<, the matrix is modified in place.

5.43.13 Modify an element or a row of a matrix : subsop

subsop modifies an element or a row of a matrix. It is used mainly for Maple and MuPAD compatibility. Unlike := or =<, it does not require the matrix to be stored in a variable.
subsop takes two or three arguments, **these arguments are permuted** in Maple mode.

1. Modify an element

   - In Xcas mode, the first index is 0
     subsop has two (resp. three) arguments: a matrix A and an equality [r,c]=v (resp. a matrix A, a list of indexes [r,c], a value v).
     subsop replaces the element A[r,c] by v.
     Input in Xcas mode :

     `subsop([[4,5],[2,6]],[1,0]=3)`

     or :

     `subsop([[4,5],[2,6]],[1,0],3)`

     Output :

     `[[4,5],[3,6]]`

     Remark
     If the matrix is stored in a variable, for example A:=[[4,5],[2,6]], it is easier to input A[1,0]:=3 which modifies A into the matrix [[4,5],[3,6]].

   - In Mupad, TI mode, the first index is 1
     subsop has two (resp. three) arguments: a matrix A and an equality [r,c]=v (resp. a matrix A, a list of index [r,c], a value v).
     subsop replaces the element A[r,c] by v.
     Input in Mupad, TI mode :

     `subsop([[4,5],[2,6]],[2,1]=3)`

     or :

     `subsop([[4,5],[2,6]],[2,1],3)`

     Output :

     `[[4,5],[3,6]]`

     Remark
     If the matrix is stored in a variable, for example A:=[[4,5],[2,6]], it is easier to input A[2,1]:=3 which modifies A into the matrix [[4,5],[3,6]].

   - In Maple mode, the arguments are permuted and the first index is 1
     subsop has two arguments: an equality [r,c]=v and a matrix A.
     subsop replaces the element A[r,c] by v.
     Input in Maple mode

     `subsop([2,1]=3,[[4,5],[2,6]])`

Output :

```
[[4,5],[3,6]]
```

Remark
If the matrix is stored in a variable, for example `A:=[[4,5],[2,6]]`, it is easier to input `A[2,1]:=3` which modifies `A` into the matrix `[[4,5],[3,6]]`.

2. Modify a row

   - in `Xcas` mode, the first index is 0
     `subsop` takes two arguments : a matrix and an equality (the index of the row to be modified, the = sign and the new row value).
     Input in `Xcas` mode :

     ```
     subsop([[4,5],[2,6]],1=[3,3])
     ```

     Output :

     ```
     [[4,5],[3,3]]
     ```

     Remark
     If the matrix is stored in a variable, for example `A:=[[4,5],[2,6]]`, is is easier to input `A[1]:=[3,3]` which modifies `A` into the matrix `[[4,5],[3,3]]`.
   - In `Mupad, TI` mode, the first index is 1
     `subsop` takes two arguments : a matrix and an equality (the index of the row to be modified, the = sign and the new row value).
     Input in `Mupad, TI` mode :

     ```
     subsop([[4,5],[2,6]],2=[3,3])
     ```

     Output :

     ```
     [[4,5],[3,3]]
     ```

     Remark
     If the matrix is stored in a variable, for example `A:=[[4,5],[2,6]]`, it is easier to input `A[2]:=[3,3]` which modifies `A` into the matrix `[[4,5],[3,3]]`.
   - in `Maple` mode, the arguments are permuted and the first index is 1 :
     `subsop` takes two arguments : an equality (the index of the row to be modified, the = sign and the new row value) and a matrix.
     Input in `Maple` mode :

     ```
     subsop(2=[3,3],[[4,5],[2,6]])
     ```

     Output :

     ```
     [[4,5],[3,3]]
     ```

     Remark
     If the matrix is stored in a variable, for example `A:=[[4,5],[2,6]]`, it is easier to input `A[2]:=[3,3]` which modifies `A` into the matrix `[[4,5],[3,3]]`.

Remark
Note also that subsop with a 'n=NULL' argument deletes row number n. In Xcas mode input :

```
subsop([[4,5],[2,6]],'1=NULL')
```

Output :

```
[[4,5]]
```

5.43.14 Extract rows or columns of a matrix (Maple compatibility) : row col

row (resp. col) extracts one or several rows (resp. columns) of a matrix.
row (resp. col) takes 2 arguments : a matrix A, and an integer n or an interval $n_1..n_2$.
row (resp. col) returns the row (resp. column) of index n of A, or the sequence of rows (resp. columns) of index from n_1 to n_2 of A.
Input :

```
row([[1,2,3],[4,5,6],[7,8,9]],1)
```

Output :

```
[4,5,6]
```

Input :

```
row([[1,2,3],[4,5,6],[7,8,9]],0..1)
```

Output :

```
([1,2,3],[4,5,6])
```

Input :

```
col([[1,2,3],[4,5,6],[7,8,9]],1)
```

Output :

```
[2,5,8]
```

Input :

```
col([[1,2,3],[4,5,6],[7,8,9]],0..1)
```

Output :

```
([1,4,7,[2,5,8])
```

5.43.15 Remove rows or columns of a matrix : `delrows delcols`

`delrows` (resp. `delcols`) removes one or several rows (resp. columns) of a matrix.
`delrows` (resp. `delcols`) takes 2 arguments : a matrix A, and an interval $n_1..n_2$.
`delrows` (resp. `delcols`) returns the matrix where the rows (resp. columns) of index from n_1 to n_2 of A are removed.
Input :

```
delrows([[1,2,3],[4,5,6],[7,8,9]],1..1)
```

Output :

```
[[1,2,3],[7,8,9]]
```

Input :

```
delrows([[1,2,3],[4,5,6],[7,8,9]],0..1)
```

Output :

```
[[7,8,9]]
```

Input :

```
delcols([[1,2,3],[4,5,6],[7,8,9]],1..1)
```

Output :

```
[[1,3],[4,6],[7,9]]
```

Input :

```
delcols([[1,2,3],[4,5,6],[7,8,9]],0..1)
```

Output :

```
[[3],[6],[9]]
```

5.43.16 Extract a sub-matrix of a matrix (TI compatibility) : `subMat`

`subMat` takes 5 arguments : a matrix A, and 4 integers $nl1, nc1, nl2, nc2$, where $nl1$ is the index of the first row, $nc1$ is the index of the first column, $nl2$ is the index of the last row and $nc2$ is the index of the last column.
`subMat(A,nl1,nc1,nl2,nc2)` extracts the sub-matrix of the matrix `A` with first element `A[nl1,nc1]` and last element `A[nl2,nc2]`.
Define the matrix `A` :

```
A:=[[3,4,5],[1,2,6]]
```

Input :

```
subMat(A,0,1,1,2)
```

Output :

```
[[4,5],[2,6]]
```

Input :

```
subMat(A,0,1,1,1]
```

Output :

```
[[4],[2]]
```

By default $nl1 = 0$, $nc1 = 0$, $nl2$=`nrows(A)`-1 and $nc2$=`ncols(A)`-1
Input :

```
subMat(A,1)
```

or :

```
subMat(A,1,0)
```

or :

```
subMat(A,1,0,1)
```

or :

```
subMat(A,1,0,1,2)
```

Output :

```
[[1,2,6]]
```

5.43.17 Add a row to another row : `rowAdd`

`rowAdd` takes three arguments : a matrix A and two integers $n1$ and $n2$.
`rowAdd` returns the matrix obtained by replacing in A, the row of index $n2$ by the sum of the rows of index $n1$ and $n2$.
Input :

```
rowAdd([[1,2],[3,4]],0,1)
```

Output :

```
[[1,2],[4,6]]
```

5.43.18 Multiply a row by an expression : `mRow`

`mRow` takes three arguments : an expression, a matrix A and an integer n.
`mRow` returns the matrix obtained by replacing in A, the row of index n by the product of the row of index n by the expression.
Input :

```
mRow(12,[[1,2],[3,4]],1)
```

Output :

```
[[1,2],[36,48]]
```

5.43.19 Add k times a row to an another row : `mRowAdd`

`mRowAdd` takes four arguments : a real k, a matrix A and two integers $n1$ and $n2$.
`mRowAdd` returns the matrix obtained by replacing in A, the row of index $n2$ by the sum of the row of index $n2$ and k times the row of index $n1$.
Input :

```
mRowAdd(1.1,[[5,7],[3,4],[1,2]],1,2)
```

Output :

```
[[5,7],[3,4],[4.3,6.4]]
```

5.43.20 Exchange two rows : `rowSwap`

`rowSwap` takes three arguments : a matrix A and two integers $n1$ and $n2$.
`rowSwap` returns the matrix obtained by exchanging in A, the row of index $n1$ with the row of index $n2$.
Input :

```
rowSwap([[1,2],[3,4]],0,1)
```

Output :

```
[[3,4],[1,2]]
```

5.43.21 Make a matrix with a list of matrices : `blockmatrix`

`blockmatrix` takes as arguments two integers n, m and a list of size $n * m$ of matrices of the same dimension $p \times q$ (or more generally such that the m first matrices have the same number of rows and c columns, the m next rows have the same number of rows and c columns, and so on ...). In both cases, we have n blocks of c columns.
`blockmatrix` returns a matrix having c columns by putting these n blocks one under another (vertical gluing). If the matrix arguments have the same dimension $p \times q$, the answer is a matrix of dimension $p * n \times q * m$.
Input :

```
blockmatrix(2,3,[idn(2),idn(2),idn(2),
        idn(2),idn(2),idn(2)])
```

Output :

```
[[1,0,1,0,1,0],[0,1,0,1,0,1],
 [1,0,1,0,1,0],[0,1,0,1,0,1]]
```

Input :

```
blockmatrix(3,2,[idn(2),idn(2),
 idn(2),idn(2),idn(2),idn(2)])
```

Output :

```
[[1,0,1,0],[0,1,0,1],
[1,0,1,0],[0,1,0,1],[1,0,1,0],[0,1,0,1]]
```

Input :

```
blockmatrix(2,2,[idn(2),newMat(2,3),
newMat(3,2),idn(3)])
```

Output :

```
[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],
[0,0,0,1,0],[0,0,0,0,1]]
```

Input :

```
blockmatrix(3,2,[idn(1),newMat(1,4),
newMat(2,3),idn(2),newMat(1,2),[[1,1,1]]])
```

Output :

```
[[1,0,0,0,0],[0,0,0,1,0],[0,0,0,0,1],[0,0,1,1,1]]
```

Input :

```
A:=[[1,1],[1,1]];B:=[[1],[1]]
```

then :

```
blockmatrix(2,3,[2*A,3*A,4*A,5*B,newMat(2,4),6*B])
```

Output :

```
[[2,2,3,3,4,4],[2,2,3,3,4,4],
[5,0,0,0,0,6],[5,0,0,0,0,6]]
```

5.43.22 Make a matrix from two matrices : semi_augment

semi_augment concat two matrices with the same number of columns.
Input :

```
semi_augment([[3,4],[2,1],[0,1]],[[1,2],[4,5]])
```

Output :

```
[[3,4],[2,1],[0,1],[1,2],[4,5]]
```

Input :

```
semi_augment([[3,4,2]],[[1,2,4]])
```

Output :

```
[[3,4,2],[1,2,4]]
```

Note the difference with concat.
Input :

```
concat([[3,4,2]],[[1,2,4]]
```

Output :

```
[[3,4,2,1,2,4]]
```

Indeed, when the two matrices A and B have the same dimension, `concat` makes a matrix with the same number of rows as A and B by gluing them side by side.
Input :

```
concat([[3,4],[2,1],[0,1]],[[1,2],[4,5]]
```

Output :

```
[[3,4],[2,1],[0,1],[1,2],[4,5]]
```

but input :

```
concat([[3,4],[2,1]],[[1,2],[4,5]]
```

Output :

```
[[3,4,1,2],[2,1,4,5]]
```

5.43.23 Make a matrix from two matrices : `augment concat`

`augment` or `concat` concats two matrices A and B having the same number of rows, or having the same number of columns. In the first case, it returns a matrix having the same number of rows as A and B by horizontal gluing, in the second case it returns a matrix having the same number of columns by vertical gluing.
Input :

```
augment([[3,4,5],[2,1,0]],[[1,2],[4,5]])
```

Output :

```
[[3,4,5,1,2],[2,1,0,4,5]]
```

Input :

```
augment([[3,4],[2,1],[0,1]],[[1,2],[4,5]])
```

Output :

```
[[3,4],[2,1],[0,1],[1,2],[4,5]]
```

Input :

```
augment([[3,4,2]],[[1,2,4]]
```

Output :

```
[[3,4,2,1,2,4]]
```

Note that if A and B have the same dimension, `augment` makes a matrix with the same number of rows as A and B by horizontal gluing, in that case you must use `semi_augment` for vertical gluing.
Input :

```
augment([[3,4],[2,1]],[[1,2],[4,5]])
```

Output :

```
[[3,4,1,2],[2,1,4,5]]]
```

5.43.24 Build a matrix with a function : `makemat`

`makemat` takes three arguments :

- a function of two variables `j` and `k` which should return the value of $a_{j,k}$, the element of row index `j` and column index `k` of the matrix to be built.
- two integers n and p.

`makemat` returns the matrix $A = (a_{j,k})$ ($j = 0..n-1$ and $k = 0..p-1$) of dimension $n \times p$.
Input :

```
makemat((j,k)->j+k,4,3)
```

or first define the h function:

```
h(j,k):=j+k
```

then, input:

```
makemat(h,4,3)
```

Output :

```
[[0,1,2],[1,2,3],[2,3,4],[3,4,5]]
```

Note that the indices are counted starting from 0.

5.43.25 Define a matrix : `matrix`

`matrix` takes three arguments :

- two integers n and p.
- a function of two variables `j` and `k` which should return the value of $a_{j,k}$, the element of row index `j` and column index `k` of the matrix to be build.

`matrix` returns the matrix $A = (a_{j,k})$ ($j = 1..n$ and $k = 1..p$) of dimension $n \times p$.
Input :

```
matrix(4,3,(j,k)->j+k)
```

or first define the h function:

```
h(j,k):=j+k
```

then, input:

```
matrix(4,3,h)
```

Output :

```
[[2,3,4],[3,4,5],[4,5,6],[5,6,7]]
```

Note the argument order and the fact that the indices are counted starting from 1. If the last argument is not provided, it defaults to 0.

5.43.26 Append a column to a matrix : `border`

`border` takes as argument a matrix `A` of dimension $p*q$ and a list `b` of size p (i.e. `nrows(A)=size(b)`).
`border` returns the matrix obtained by appending `tran(b)` as last column to the matrix `A`, therefore:

```
border(A,b)=tran([op(tran(A)),b])=tran(append(tran(A),b))
```

Input :

```
border([[1,2,4],[3,4,5]],[6,7])
```

Output :

```
[[1,2,4,6],[3,4,5,7]]
```

Input :

```
border([[1,2,3,4],[4,5,6,8],[7,8,9,10]],[1,3,5])
```

Output :

```
[[1,2,3,4,1],[4,5,6,8,3],[7,8,9,10,5]]
```

5.43.27 Count the elements of a matrix verifying a property : `count`

`count` takes as arguments : a real function `f` and a real matrix `A` of dimension `p*q` (resp. a list `l` of size `n`).
`count` returns `f(A[0,0])+..f(A[p-1,q-1])` (resp. `f(l[0])+..f(l[n-1])`)
Hence, if `f` is a boolean function, `count` returns the number of elements of the matrix `A` (resp. the list `l`) verifying the property `f`.
Input :

```
count(x->x,[[2,12],[45,3],[7,78]])
```

Output :

```
147
```

indeed: 2+12+45+3+7+78=147.
Input :

```
count(x->x<10,[[2,12],[45,3],[7,78]])
```

Output :

```
3
```

5.43.28 Count the elements equal to a given value : `count_eq`

`count_eq` takes as arguments: a real and a real list or a real matrix.
`count_eq` returns the number of elements of the list or matrix equal to the first argument.
Input :

```
count_eq(12,[[2,12,45],[3,7,78]])
```

Output :

```
1
```

5.43.29 Count the elements smaller than a given value : count_inf

count_inf takes as arguments: a real and a real list or a real matrix.
count_inf returns the number of elements of the list or matrix which are strictly less than the first argument.
Input :

```
count_inf(12,[2,12,45,3,7,78])
```

Output :

```
3
```

5.43.30 Count the elements greater than a given value : count_sup

count_sup takes as arguments: a real and a real list or a real matrix.
count_sup returns the number of elements of the list or matrix which are strictly greater to the first argument.
Input :

```
count_sup(12,[[2,12,45],[3,7,78]])
```

Output :

```
2
```

5.43.31 Statistics functions acting on column matrices : mean, stddev, variance, median, quantile, quartiles, boxwhisker

The following functions work on matrices, acting column by column:

- mean computes the arithmetic means of the statistical series stored in the columns of a matrix.
  Input :

  ```
  mean([[3,4,2],[1,2,6]])
  ```

  Output is the vector of the means of each column :

  ```
  [2,3,4]
  ```

  Input :

  ```
  mean([[1,0,0],[0,1,0],[0,0,1]])
  ```

  Output

  ```
  [1/3,1/3,1/3]
  ```

- stddev computes the standard deviations of the population statistical series stored in the columns of a matrix.
  Input :

```
stddev([[3,4,2],[1,2,6]])
```

Output is the vector of the standard deviations of each column :

```
[1,1,2]
```

- `variance` computes the variances of the statistical series stored in the columns of a matrix.
  Input :

```
variance([[3,4,2],[1,2,6]])
```

Output is the vector of the variance of each column :

```
[1,1,4]
```

- `median` computes the medians of the statistical series stored in the columns of a matrix.
  Input :

```
median([[6,0,1,3,4,2,5],[0,1,3,4,2,5,6],[1,3,4,2,5,6,0],
 [3,4,2,5,6,0,1],[4,2,5,6,0,1,3],[2,5,6,0,1,3,4]])
```

Output is the vector of the median of each column :

```
[3,3,4,4,4,3,4]
```

- `quantile` computes the deciles as specified by the second argument of the statistical series stored in the columns of a matrix.
  Input :

```
quantile([[6,0,1,3,4,2,5],[0,1,3,4,2,5,6],[1,3,4,2,5,6,0],
[3,4,2,5,6,0,1],[4,2,5,6,0,1,3],[2,5,6,0,1,3,4]],0.25)
```

Output is the vector of the first quartile of each column :

```
[1,1,2,2,1,1,1]
```

Input :

```
quantile([[6,0,1,3,4,2,5],[0,1,3,4,2,5,6],[1,3,4,2,5,6,0],
[3,4,2,5,6,0,1],[4,2,5,6,0,1,3],[2,5,6,0,1,3,4]],0.75)
```

Output is the vector of the third quartile of each column :

```
[3,3,4,4,4,3,4]
```

- `quartiles` computes the minima, the first quartiles, the medians, the third quartiles and the maxima of the statistical series stored in the columns of a matrix.
  Input :

```
quartiles([[6,0,1,3,4,2,5],[0,1,3,4,2,5,6],[1,3,4,2,5,6,0],
         [3,4,2,5,6,0,1], [4,2,5,6,0,1,3],
                [2,5,6,0,1,3,4]])
```

  Output is a matrix, its first row is the minima of each column, its second row is the fist quartiles of each column, its third row the medians of each column, its fourth row the third quartiles of each column and its last row the maxima of each column:

```
[[0,0,1,0,0,0,0],[1,1,2,2,1,1,1], [2,2,3,3,2,2,3],

         [3,3,4,4,4,3,4],[6,5,6,6,6,6,6]]
```

- `boxwhisker` draws the whisker boxes of the statistical series stored in the columns of a matrix .
  Input :

```
boxwhisker([[6,0,1,3,4,2,5],[0,1,3,4,2,5,6],
      [1,3,4,2,5,6,0],[3,4,2,5,6,0,1],
     [4,2,5,6,0,1,3],[2,5,6,0,1,3,4]])
```

  Output :

```
the drawing of the whisker boxes of the
statistical series of each column of the matrix
argument
```

5.43.32 Dimension of a matrix : `dim`

`dim` takes as argument a matrix A.
`dim` returns the list of the number of rows and columns of the matrix A.
Input :

```
dim([[1,2,3],[3,4,5]])
```

Output :

```
[2,3]
```

5.43.33 Number of rows : `rowdim rowDim nrows`

`rowdim` (or `rowDim` or `nrows`) takes as argument a matrix A.
`rowdim` (or `rowDim` or `nrows`) returns the number of rows of the matrix A.
Input :

```
rowdim([[1,2,3],[3,4,5]])
```

or :

```
nrows([[1,2,3],[3,4,5]])
```

Output :

```
2
```

5.43.34 Number of columns : `coldim colDim ncols`

`coldim` (or `colDim` or `ncols`) takes as argument a matrix A.
`coldim` (or `colDim` or `ncols`) returns the number of columns of the matrix A.
Input :

```
coldim([[1,2,3],[3,4,5]])
```

or :

```
ncols([[1,2,3],[3,4,5]])
```

Output :

```
3
```

5.44 Linear algebra

5.44.1 Transpose of a matrix : `tran transpose`

`tran` or `transpose` takes as argument a matrix A.
`tran` or `transpose` returns the transpose matrix of A.
Input :

```
tran([[1,2],[3,4]])
```

Output :

```
[[1,3],[2,4]]
```

5.44.2 Inverse of a matrix : `inv /`

`inv` takes as argument a square matrix A.
`inv` returns the inverse matrix of A.
Input :

```
inv([[1,2],[3,4]])
```

or :

```
1/[[1,2],[3,4]])
```

or :

```
A:=[[1,2],[3,4]];1/A
```

Output :

```
[[-2,1],[3/2,1/-2]]
```

5.44.3 Trace of a matrix : `trace`

`trace` takes as argument a matrix A.
`trace` returns the trace of the matrix A, that is the sum of the diagonal elements.
Input :

```
trace([[1,2],[3,4]])
```

Output :

```
5
```

5.44.4 Determinant of a matrix : `det`

`det` takes as argument a matrix A.
`det` returns the determinant of the matrix A.
Input :

```
det([[1,2],[3,4]])
```

Output :

```
-2
```

Input :

```
det(idn(3))
```

Output :

```
1
```

5.44.5 Determinant of a sparse matrix : `det_minor`

`det_minor` takes as argument a matrix A.
`det_minor` returns the determinant of the matrix A computed by expanding the determinant using Laplace's algorithm.
Input :

```
det_minor([[1,2],[3,4]])
```

Output :

```
-2
```

Input :

```
det_minor(idn(3))
```

Output :

```
1
```

5.44.6 Rank of a matrix : `rank`

`rank` takes as argument a matrix A.
`rank` returns the rank of the matrix A.
Input :

```
rank([[1,2],[3,4]])
```

Output :

```
2
```

Input :

```
rank([[1,2],[2,4]])
```

Output :

```
1
```

5.44.7 Transconjugate of a matrix : `trn`

`trn` takes as argument a matrix A.
`trn` returns the transconjugate of A (i.e. the conjugate of the transpose matrix of A).
Input :

```
trn([[i, 1+i],[1, 1-i]])
```

Output after simplification:

```
[[-i,1],[1-i,1+i]]
```

5.44.8 Equivalent matrix : `changebase`

`changebase` takes as argument a matrix A and a change-of-basis matrix P.
`changebase` returns the matrix B such that $B = P^{-1}AP$.
Input :

```
changebase([[1,2],[3,4]],[[1,0],[0,1]])
```

Output :

```
[[1,2],[3,4]]
```

Input :

```
changebase([[1,1],[0,1]],[[1,2],[3,4]])
```

Output :

```
[[-5,-8],[9/2,7]]
```

Indeed :

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} * \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -8 \\ \frac{9}{2} & 7 \end{bmatrix}$$

.

5.44.9 Basis of a linear subspace : `basis`

`basis` takes as argument a list of vectors generating a linear subspace of $\mathbb{R}^n$.
`basis` returns a list of vectors, that is a basis of this linear subspace.
Input :

```
basis([[1,2,3],[1,1,1],[2,3,4]])
```

Output :

```
[[1,0,-1], [0,1,2]]
```

5.44.10 Basis of the intersection of two subspaces : `ibasis`

`ibasis` takes as argument two lists of vectors generating two subspaces of $\mathbb{R}^n$.
`ibasis` returns a list of vectors, that is a basis of the intersection of these two subspaces.
Input :

```
ibasis([[1,2]],[[2,4]])
```

Output :

```
[[1,2]]
```

5.44.11 Image of a linear function : `image`

`image` takes as argument the matrix of a linear function f with respect to the canonical basis.
`image` returns a list of vectors that is a basis of the image of f.
Input :

```
image([[1,1,2],[2,1,3],[3,1,4]])
```

Output :

```
[[-1,0,1],[0,-1,-2]]
```

5.44.12 Kernel of a linear function : `kernel nullspace ker`

`ker` (or `kernel` or `nullspace`) takes as argument the matrix of an linear function f with respect to the canonical basis.
`ker` (or `kernel` or `nullspace`) returns a list of vectors that is a basis of the kernel of f.
Input :

```
ker([[1,1,2],[2,1,3],[3,1,4]])
```

Output :

```
[[1,1,-1]]
```

The kernel is generated by the vector `[1,1,-1]`.

5.44.13 Kernel of a linear function : `Nullspace`

Warning This function is useful in Maple mode only (hit the state line red button then `Prog style`, then choose Maple and Apply).
`Nullspace` is the inert form of `nullspace`.
`Nullspace` takes as argument an integer matrix of a linear function f with respect to the canonical basis.
`Nullspace`) followed by `mod p` returns a list of vectors that is a basis of the kernel of f computed in $\mathbb{Z}/p\mathbb{Z}[X]$.
Input :

```
Nullspace([[1,1,2],[2,1,3],[3,1,4]])
```

Output :

```
nullspace([[1,1,2],[2,1,3],[3,1,4]])
```

Input (in Maple mode):

```
Nullspace([[1,2],[3,1]]) mod 5
```

Output :

```
[2,-1]
```

In Xcas mode, the equivalent input is :

```
nullspace([[1,2],[3,1]] % 5)
```

Output :

```
[2% 5,-1]
```

5.44.14 Subspace generated by the columns of a matrix : `colspace`

`colspace` takes as argument the matrix A of a linear function f with respect to the canonical basis.
`colspace` returns a matrix. The columns of this matrix are a basis of the subspace generated by the columns of A.
`colspace` may have a variable name as second argument, where `Xcas` will store the dimension of the subspace generated by the columns of A.
Input :

```
colspace([[1,1,2],[2,1,3],[3,1,4]])
```

Output :

```
[[-1,0],[0,-1],[1,-2]]
```

Input :

```
colspace([[1,1,2],[2,1,3],[3,1,4]],dimension)
```

Output :

```
[[-1,0],[0,-1],[1,-2]]
```

Then input:

```
dimension
```

Output :

```
2
```

5.44.15 Subspace generated by the rows of a matrix : `rowspace`

`rowspace` takes as argument the matrix A of a linear function f with respect to the canonical basis.
`rowspace` returns a list of vectors that is a basis of the subspace generated by the rows of A.
`rowspace` may have a variable name as second argument where `Xcas` will store the dimension of the subspace generated by the rows of A.
Input :

```
rowspace([[1,1,2],[2,1,3],[3,1,4]])
```

Output :

```
[[-1,0,-1],[0,-1,-1]]
```

Input :

```
rowspace([[1,1,2],[2,1,3],[3,1,4]],dimension)
```

Output :

```
[[-1,0,-1],[0,-1,-1]]
```

Then input:

```
dimension
```

Output :

```
2
```

5.45 Linear Programmation

Linear programming problems are maximization problem of a linear functionals under linear equality or inequality constraints. The most simple case can be solved directly by the so-called simplex algorithm. Most cases require to solve an auxiliary linear programming problem to find an initial vertex for the simplex algorithm.

5.45.1 Simplex algorithm: `simplex_reduce`

The simple case
The function `simplex_reduce` makes the reduction by the simplex algorithm to find :

$$\max(c.x), \quad A.x \leq b, \ x \geq 0, \ b \geq 0$$

where c, x are vectors of $\mathbb{R}^n$, $b \geq 0$ is a vector in $\mathbb{R}^p$ and A is a matrix of p rows and n columns.
`simplex_reduce` takes as argument `A,b,c` and returns `max(c.x)`, the augmented solution of `x` (augmented since the algorithm works by adding rows(A) auxiliary variables) and the reduced matrix.

Example
Find

$$\max(X+2Y) \text{ where } \begin{cases} (X,Y) & \geq & 0 \\ -3X+2Y & \leq & 3 \\ X+Y & \leq & 4 \end{cases}$$

Input :

```
simplex_reduce([[-3,2],[1,1]],[3,4],[1,2])
```

Output :

```
7,[1,3,0,0],[[0,1,1/5,3/5,3],[1,0,(-1)/5,2/5,1],
                    [0,0,1/5,8/5,7]]
```

Which means that the maximum of `X+2Y` under these conditions is `7`, it is obtained for `X=1,Y=3` because `[1,3,0,0]` is the augmented solution and the reduced matrix is :
`[[0,1,1/5,3/5,3],[1,0,(-1)/5,2/5,1], [0,0,1/5,8/5,7]]`.

A more complicated case that reduces to the simple case
With the former call of `simplex_reduce`, we have to :

- rewrite constraints to the form $x_k \geq 0$,
- remove variables without constraints,
- add variables such that all the constraints have positive components.

For example, find :

$$\min(2x+y-z+4) \quad \text{where} \begin{cases} x & \leq & 1 \\ y & \geq & 2 \\ x+3y-z & = & 2 \\ 2x-y+z & \leq & 8 \\ -x+y & \leq & 5 \end{cases} \tag{5.1}$$

Let $x = 1 - X$, $y = Y + 2$, $z = 5 - X + 3Y$ the problem is equivalent to finding the minimum of $(-2X + Y - (5 - X + 3Y) + 8)$ where :

$$\begin{cases} X & \geq & 0 \\ Y & \geq & 0 \\ 2(1-X)-(Y+2)+5-X+3Y & \leq & 8 \\ -(1-X)+(Y+2) & \leq & 5 \end{cases}$$

or to find the minimum of :

$$(-X-2Y+3) \quad \text{where} \begin{cases} X & \geq & 0 \\ Y & \geq & 0 \\ -3X+2Y & \leq & 3 \\ X+Y & \leq & 4 \end{cases}$$

i.e. to find the maximum of $-(-X-2Y+3) = X+2Y-3$ under the same conditions, hence it is the same problem as to find the maximum of $X+2Y$ seen before. We found `7`, hence, the result here is `7-3=4`.

The general case

A linear programming problem may not in general be directly reduced like above to the simple case. The reason is that a starting vertex must be found before applying the simplex algorithm. Therefore, `simplex_reduce` may be called by specifying this starting vertex, in that case, all the arguments including the starting vertex are grouped in a single matrix.

We first illustrate this kind of call in the simple case where the starting point does not require solving an auxiliary problem. If `A` has p rows and n columns and if we define :

```
B:=augment(A,idn(p)); C:=border(B,b);
d:=append(-c,0$(p+1)); D:=augment(C,[d]);
```

`simplex_reduce` may be called with `D` as single argument.
For the previous example, input :

```
A:=[[-3,2],[1,1]];B:=augment(A,idn(2));
C:=border(B,[3,4]); D:=augment(C,[[-1,-2,0,0,0]])
```

Here `C=[[-3,2,1,0,3],[1,1,0,1,4]]`
and `D=[[-3,2,1,0,3],[1,1,0,1,4],[-1,-2,0,0,0]]`
Input :

```
simplex_reduce(D)
```

Output is the same result as before.

Back to the general case.

The standard form of a linear programming problem is similar to the simplest case above, but with $Ax = b$ (instead of $Ax \leq b$) under the conditions $x \geq 0$. We may further assume that $b \geq 0$ (if not, one can change the sign of the corresponding line).

- The first problem is to find an x in the $Ax = b, x \geq 0$ domain. Let m be the number of lines of A. Add artificial variables $y_1, ..., y_m$ and maximize $-\sum y_i$ under the conditions $Ax = b, x \geq 0, y \geq 0$ starting with initial value 0 for x variables and $y = b$ (to solve this with `Xcas`, call `simplex_reduce` with a single matrix argument obtained by augmenting A by the identity, b unchanged and an artificial c with 0 under A and 1 under the identity). If the maximum exists and is 0, the identity submatrix above the last column corresponds to an x solution, we may forget the artificial variables (they are 0 if the maximum is 0).

- Now we make a second call to `simplex_reduce` with the original c and the value of x we found in the domain.

- Example : find the minimum of $2x + 3y - z + t$ with $x, y, z, t \geq 0$ and :

$$\left\{ \begin{array}{rcl} -x - y + t &=& 1 \\ y - z + t &=& 3 \end{array} \right.$$

  This is equivalent to find the opposite of the maximum of $-(2x+3y-z+t)$. Let us add two artificial variables y_1 and y_2,

```
simplex_reduce([[-1,-1,0,1,1,0,1],
[0,1,-1,1,0,1,3],
[0,0,0,0,1,1,0]])
```

Output: optimum=0, artificial variables=0, and the matrix

$$\begin{pmatrix} -1/2 & 0 & -1/2 & 1 & 1/2 & 1/2 & 2 \\ 1/2 & 1 & -1/2 & 0 & -1/2 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Columns 2 and 4 are the columns of the identity (in lines 1 and 2). Hence $x = (0, 1, 0, 2)$ is an initial point in the domain. We are reduced to solve the initial problem, after replacing the lines of $Ax = b$ by the two first lines of the answer above, removing the last columns corresponding to the artificial variables. We add $c.x$ as last line

```
simplex_reduce([[-1/2,0,-1/2,1,2],
[1/2,1,-1/2,0,1],[2,3,-1,1,0]])
```

Output: maximum=-5, hence the minimum of the opposite is 5, obtained for $(0, 1, 0, 2)$, after replacement $x = 0$, $y = 1$, $z = 0$ and $t = 2$.

For more details, search google for `simplex algorithm`.

5.45.2 Solving general linear programming problems: `lpsolve`

Linear programming problems (where a multivariate linear function needs to be maximized or minimized subject to linear (in)equality constraints), as well as (mixed) integer programming problems, can be solved by using the function `lpsolve`. Problems can be entered directly (in symbolic or matrix form) or loaded from a file in LP or (gzipped) MPS format.

`lpsolve` accepts four arguments :

1. `obj` : symbolic expression representing the objective function or path to file containing LP problem (in the latter case parameter `constr` should not be given)

2. `constr` (optional) : list of linear constraints which may be equalities or inequalities or bounded expressions entered as `expr=a..b`

3. `bd` (optional) : sequence of expressions of type `var=a..b` specifying that the variable `var` is bounded with `a` below and with `b` above

4. `opts` (optional) : sequence of solver settings in form `option=value`, where `option` may be one of :

   `assume` – one of `lp_nonnegative`, `lp_integer` (`integer`), `lp_binary` or `lp_nonnegint` (or `nonnegint`), default : *unset*

   `lp_integervariables` – list of identifiers or indices (of integer variables), default : *empty* lp_binaryvariables – list of identifiers or indices (of binary variables), default : *empty*

lp_maximize (or maximize) – *true* or *false* (objective direction), default : *false*

lp_method – one of exact, float, lp_simplex or lp_interiorpoint (solver type), default lp_simplex

lp_depthlimit – positive integer (max. depth of branch&bound tree), default : *unlimited*

lp_nodelimit – positive integer (max. nodes in branch&bound tree), default : *unlimited*

lp_iterationlimit – positive integer (max. iterations of simplex algorithm), default : *unlimited*

lp_timelimit – positive number (max. solving time in milliseconds), default : *unlimited*

lp_maxcuts – nonnegative integer (max. GMI cuts per node), default : 5

lp_gaptolerance – positive number (relative integrality gap threshold), default : 0

lp_nodeselect – one of lp_depthfirst, lp_breadthfirst, lp_hybrid or lp_bestprojection (branching node selection strategy), default : lp_hybrid

lp_varselect – one of lp_firstfractional, lp_lastfractional, lp_mostfractional or lp_pseudocost (branching variable selection strategy), default : lp_pseudocost

lp_verbose – *true* or *false*, default : *false*

The return value is in the form [optimum,soln] where optimum is the minimum/maximum value of the objective function and soln is the list of coordinates corresponding to the point at which the optimal value is attained, i.e. the optimal solution. If there is no feasible solution, an empty list is returned. When the objective function is unbounded, optimum is returned as +infinity (for maximization problems) or -infinity (for minimization problems). If an error is experienced while solving (terminating the process), undef is returned.

The given objective function is minimized by default. To maximize it, include the option lp_maximize=true or lp_maximize or simply maximize. Also note that all variables are, unless specified otherwise, assumed to be continuous and unrestricted in sign.

Solving LP problems

By default, lpsolve uses primal simplex method implementation to solve LP problems. For example, to solve the problem specified in (5.1), input :

```
constr:=[x<=1,y>=2,x+3y-z=2,3x-y+z<=8,-x+y<=5];
            lpsolve(2x+y-z+4,constr)
```

Output :

```
[-4,[x=0,y=5,z=13]]
```

Therefore, the minimum value of $f(x,y,z) = 2\,x + y - z + 4$ is equal to -4 under the given constraints. The optimal value is attained at point $(x,y,z) = (0,5,13)$.

Constraints may also take the form `expr=a..b` for bounded linear expressions.
Input :

```
lpsolve(x+2y+3z,[x+y=1..5,y+z+1=2..4,x>=0,y>=0])
```

Output :

```
[-2,[x=0,y=5,z=-4]]
```

Use the `assume=lp_nonnegative` option to specify that all variables are nonnegative. It is easier than entering the nonnegativity constraints explicitly.
Input:

```
lpsolve(-x-y,[y<=3x+1/2,y<=-5x+2],
        assume=lp_nonnegative)
```

Output:

```
[-5/4,[x=3/16,y=17/16]]
```

Bounds can be added separately for some variables. They should be entered after constraints.
Input :

```
constr:=[5x-10y<=20,2z-3y=6,-x+3y<=3];
lpsolve(-6x+4y+z,constr,x=1..20,y=0..inf)
```

Output :

```
[-133/2,[x=18,y=7,z=27/2]]
```

Number of iterations can be limited by setting `lp_iterationlimit` to some positive integer. If maximum number of iterations is reached, the current feasible solution (not necessarily an optimal one) is returned.

Entering problems in matrix form

`lpsolve` supports entering linear programming problems in matrix form, where `obj` is a vector of coefficients $\mathbf{c}$ and `constr` is a list $[\mathbf{A}, \mathbf{b}, \mathbf{A}_{eq}, \mathbf{b}_{eq}]$ such that objective function $\mathbf{c}^T\,\mathbf{x}$ is to be minimized/maximized subject to constraints $\mathbf{A}\,\mathbf{x} \leq \mathbf{b}$ and $\mathbf{A}_{eq}\,\mathbf{x} = \mathbf{b}_{eq}$. If a problem does not contain equality constraints, parameters $\mathbf{A}_{eq}$ and $\mathbf{b}_{eq}$ may be omitted. For a problem that does not contain inequality constraints, empty lists must be entered in place of $\mathbf{A}$ and in place of $\mathbf{b}$.

The parameter `bd` is entered as a list of two vectors $\mathbf{b}_l$ and $\mathbf{b}_u$ of the same length as the vector $\mathbf{c}$ such that $\mathbf{b}_l \leq \mathbf{x} \leq \mathbf{b}_u$. These vectors may contain `+infinity` or `-infinity`.
Input :

```
c:=[-2,1];A:=[[-1,1],[1,1],[-1,0],[0,-1]];
b:=[3,5,0,0];lpsolve(c,[A,b])
```

Output :

```
[-10,[5,0]]
```

Input :

```
c:=[-2,5,-3];bl:=[2,3,1];bu:=[6,10,7/2];
lpsolve(c,[],[bl,bu])
```

Output :

```
[-15/2,[6,3,7/2]]
```

Input :

```
c:=[4,5];Aeq:=[[-1,3/2],[-3,2]];beq:=[2,3];
lpsolve(c,[[],[],Aeq,beq])
```

Output :

```
[26/5,[-1/5,6/5]]
```

Solving MIP (Mixed Integer Programming) problems

`lpsolve` allows restricting (some) variables to integer values. Such problems, called (*mixed*) *integer programming problems*, are solved by applying branch&bound method.

To solve pure integer programming problems, in which all variables are integers, use option `assume=integer` or `assume=lp_integer`.
Input :

```
lpsolve(-5x-7y,[7x+y<=35,-x+3y<=6],assume=integer)
```

Output :

```
[-41,[x=4,y=3]]
```

Use option `assume=lp_binary` to specify that all variables are binary, i.e. the only allowed values are 0 and 1. These usually represent `false` and `true`, respectively, giving the variable a certain meaning in logical context.
Input :

```
lpsolve(8x1+11x2+6x3+4x4,[5x1+7x2+4x3+3x4<=14],
assume=lp_binary,maximize)
```

Output :

```
[21,[x1=0,x2=1,x3=1,x4=1]]
```

To solve mixed integer problems, where some variables are integers and some are continuous, use option keywords `lp_integervariables` to specify integer variables and/or `lp_binaryvariables` to specify binary variables.
Input :

```
lpsolve(x+3y+3z,[x+3y+2z<=7,2x+2y+z<=11],
    assume=lp_nonnegative,lp_maximize,
        lp_integervariables=[x,z])
```

Output :

```
[10,[x=1,y=0,z=3]]
```

Use the `assume=lp_nonnegint` or `assume=nonnegint` option to get nonnegative integer values.
Input :

```
lpsolve(2x+5y,[3x-y=1,x-y<=5],assume=nonnegint)
```

Output :

```
[12,[x=1,y=2]]
```

When specifying MIP problems in matrix form, lists corresponding to options `lp_integervariables` and `lp_binaryvariables` are populated with variable indices, like in the following example.
Input :

```
c:=[2,-3,-5];A:=[[-5,4,-5],[2,5,7],[2,-3,4]];
b:=[3,1,-2];lpsolve(c,[A,b],lp_integervariables=[0,2])
```

Output :

```
[19,[1,3/4,-1]]
```

One can also specify a range of indices instead of a list when there is too much variables. Example : `lp_binaryvariables=0..99` means that all variables x_i such that $0 \leq i \leq 99$ are binary.

Implementation details. Branch&bound algorithm by definition generates a binary tree of subproblems by branching on integer variables with fractional values. `lpsolve` features an implementation which stores only active nodes of branch&bound tree in a list, thus saving a lot of space. Also, since variable bounds are the only parameters that change during branch&bound algorithm, number of constraints does not rise with depth, which is the benefit of the upper-bounding technique built in the simplex algorithm. Therefore a steady speed and minimal resource usage is always maintained, no matter how long the execution time is. This allows for solving problems that require tens or hundreds of thousands of nodes to be generated before finding an optimal solution.

Stopping criteria. There are several ways to force the branch&bound algorithm to stop prematurely when the execution takes too much time. One can set `lp_timelimit` to integer number which defines the maximum number of milliseconds allowed to find an optimal solution. Other ways are to set `lp_nodelimit` or `lp_depthlimit` to limit the number of nodes generated in branch&bound tree or its depth, respectively. Finally, one can set `lp_gaptolerance` to some positive value, say $t > 0$, which terminates the algorithm after finding an incumbent solution and proving that the corresponding objective value differs from optimum value for less than $t \cdot 100\,\%$. It is done by monitoring the size of integrality gap, i.e. the difference between current incumbent objective value and the best objective value bound among active nodes.

If branch&bound algorithm terminates prematurely, a warning message indicating the cause is displayed. Incumbent solution, if any, is returned as the result, else the problem is declared to be infeasible.

Branching strategies. At every iteration of branch&bound algorithm, a node must be selected for branching on some variable that has a fractional optimal value for the corresponding relaxed subproblem. There exist different methods for making such decisions, called *branching strategies*. Two types of branching strategies exist: *node selection* and *variable selection* strategy.

Node selection strategy can be set by using the `lp_nodeselect` option. Possible values are :

`lp_breadthfirst` – choose the active node which provides the best bound for the objective value,

`lp_depthfirst` – choose the deepest active node and break ties by selecting the node providing the best bound,

`lp_hybrid` – combine the above two strategies,

`lp_bestprojection` – choose the node with best simple projection.

By default, `lp_bestprojection` strategy is used. Another sophisticated strategy is `lp_hybrid`: before an incumbent solution is found, solver uses `lp_depthfirst` strategy, "diving" into the tree as an incumbent solution is more likely to be located deeply. When an incumbent is found, solver switches to `lp_breadthfirst` strategy trying to close the integrality gap as quickly as possible.

Variable selection strategy can be set by using the `lp_varselect` option. Possible values are :

`lp_firstfractional` – choose the first fractional variable,

`lp_lastfractional` – choose the last fractional variable,

`lp_mostfractional` – choose the variable with fractional part closest to 0.5,

`lp_pseudocost` – choose the variable which had the greatest impact on the objective value in previous branchings.

By default, `lp_pseudocost` strategy is used. However, since pseudocost-based choice cannot be made before all integer variables have been branched upon at least one time in each direction, `lp_mostfractional` strategy is used until that condition is fulfilled.

Using the right combination of branching strategies may significantly reduce the number of subproblems needed to be examined when solving a particular MIP problem. However, what is "right" varies from problem to problem. Default strategies are the most sophisticated (as they use the available data most extensively) and usually the most effective ones. But that is not always the case, as illustrated by the following example :

Minimize $\mathbf{c}^T\,\mathbf{x}$ subject to $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, where $\mathbf{x} \in \mathbb{Z}_+^8$ and

$$\mathbf{A} = \begin{bmatrix} 22 & 13 & 26 & 33 & 21 & 3 & 14 & 26 \\ 39 & 16 & 22 & 28 & 26 & 30 & 23 & 24 \\ 18 & 14 & 29 & 27 & 30 & 38 & 26 & 26 \\ 41 & 26 & 28 & 36 & 18 & 38 & 16 & 26 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7872 \\ 10466 \\ 11322 \\ 12058 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 10 \\ 13 \\ 17 \\ 7 \\ 5 \\ 7 \\ 3 \end{bmatrix}.$$

When using the default settings, about 24000 subproblems need to be examined before an optimal solution is found. When `lp_nodeselect` is set to `lp_breadthfirst` the solver needs to examine only about 20000 subproblems, but when set to `lp_hybrid` (a strategy which in general performs better) it examines about 111000 nodes in total.

Cutting planes. Strong Gomory mixed integer cuts are generated at every node of the branch&bound tree and used to improve the objective value bound. After solving the relaxed subproblem with simplex method, at most one strong cut is generated and added to the subproblem which is subsequently reoptimized. Simplex reoptimizations are fast because they start with the last feasible basis, but applying cuts makes the simplex tableau larger, hence applying many of them may actually slow the computation down. To limit the number of cuts that can be applied to a subproblem, one can use `lp_maxcuts` option, setting it either to zero (which disables cut generation altogether) or to some positive integer. Also, one may set it to `+infinity`, which means that any number of cuts may be applied to any node. By default, `lp_maxcuts` equals to 5.

Displaying detailed output. By typing `lp_verbose=true` or simply `lp_verbose` when specifying options for `lpsolve`, detailed messages are printed during and after solving a MIP problem. During branch&bound algorithm a status report in form

```
<n>: <m> nodes active, lower bound: <lb>[, integrality gap: <g>]
```

is displayed every 5 seconds, where `n` is the number of already examined subproblems. Also, a report is printed every time incumbent solution is found or updated, as well as when the solver switches to pseudocost-based branching. After the algorithm is finished, i.e. when an optimal solution is found, summary is displayed containing the total number of examined subproblems, the number of most nodes being active at the same time and the number of applied Gomory mixed integer cuts.

In the following example, two nonnegative integers x_1 and x_2 are found such that $1867\,x_1 + 1913\,x_2 = 3618894$ and $x_1 + x_2$ is minimal. The solver shows all progress and summary messages.
Input :

```
lpsolve(x1+x2,[1867x1+1913x2=3618894],
  assume=nonnegint,lp_verbose=true)
```

Output :

```
Optimizing...
Applying branch&bound method to find integer feasible solutions...
    3937: Incumbent solution found
Summary:
 * 3938 subproblem(s) examined
 * max. tree size: 1 nodes
 * 0 Gomory cut(s) applied
```

```
[1916,[x1=1009,x2=907]]
```

Solving problems in floating-point arithmetic

`lpsolve` provides, in addition to its own exact solver implementing primal simplex method with upper-bounding technique, an interface to GLPK (GNU Linear Programming Kit) library which contains sophisticated LP/MIP solvers in floating-point arithmetic, designed to be very fast and to handle large problems. Choosing between the available solvers is done by setting `lp_method` option.

By default, `lp_method` is set to `lp_simplex`, which solves the problem using primal simplex method, but performing exact computation only when all problem coefficients are exact. If at least one of them is approximative (a floating-point number), GLPK solver is used instead (see below).

Setting `lp_method` to `exact` forces the solver to perform exact computation even when some coefficients are inexact (they are converted to rational equivalents before applying simplex method).

Specifying `lp_method=float` forces `lpsolve` to use floating-point solver. If a MIP problem is given, it is combined with branch&cut algorithm. GLPK simplex solver parameters can be controlled by setting `lp_timelimit`, `lp_gaptolerance` and `lp_varselect` options. If the latter is not set, Driebeek–Tomlin heuristic is used by default (see GLPK manual for details). If `lp_maxcuts` is greater than zero, GMI and MIR cut generation is enabled, else it is disabled. If the problem contains binary variables, cover and clique cut generation is enabled, else it is disabled. Finally, `lp_verbose=true` enables detailed messages.

Setting `lp_method` to `lp_interiorpoint` uses primal-dual interior-point algorithm which is part of GLPK. The only parameter that can be controlled via options is the verbosity level.

For example, try to solve the following LP problem using the default settings.

$$\text{Minimize } 1.06\,x_1 + 0.56\,x_2 + 3.0\,x_3$$

subject to

$$\begin{aligned} 1.06\,x1 + 0.015\,x3 &\geq 729824.87 \\ 0.56\,x_2 + 0.649\,x_3 &\geq 1522188.03 \\ x_3 &\geq 1680.05 \\ x_k &\geq 0 \quad \text{for } k = 1, 2, 3 \end{aligned}$$

Input :

```
lpsolve(1.06x1+0.56x2+3x3,[1.06x1+0.015x3>=729824.87,
       0.56x2+0.649x3>=1522188.03,x3>=1680.05],
                assume=lp_nonnegative)
```

Output :

```
[2255937.4968,[x1=688490.254009,x2=2716245.85277,x3=1680.05]]
```

If `assume=nonnegint` is used for the same problem, i.e. when $x_k \in \mathbb{Z}_+$ for $k = 1, 2, 3$, the following result is obtained by GLPK MIP solver :

```
[2255940.66,[x1=688491.0,x2=2716245.0,x3=1681.0]]
```

The solution of the original problem can also be obtained with interior-point solver by including `lp_method=lp_interiorpoint` after `assume=lp_nonnegative` :

```
[2255937.50731,[x1=688490.256652,x2=2716245.85608,
                   x3=1680.05195065]]
```

Loading problem from a file

Linear (integer) programming problems can be loaded from MPS or CPLEX LP format files (these formats are described in GLPK manual, Appendices B and C). The file name string needs to be passed as `obj` parameter. If the file name has extension "lp", CPLEX LP format is assumed, and if the extension is "mps" or "gz", MPS or gzipped MPS format is assumed.

For example, assume that `somefile.lp` file is stored in directory `/path/to/file` contains the following lines of text :

```
Maximize
obj: x1 + 2 x2 + 3 x3 + x4
Subject To
c1: - x1 + x2 + x3 + 10 x4 <= 20
c2: x1 - 3 x2 + x3 <= 30
c3: x2 - 3.5 x4 = 0
Bounds
0 <= x1 <= 40
2 <= x4 <= 3
End
```

To find an optimal solution to linear program specified in the file, one just needs to input :

```
lpsolve("/path/to/file/somefile.lp")
```

Output :

```
Reading problem data from '/path/to/file/somefile.lp'...
3 rows, 4 columns, 9 non-zeros
10 lines were read

              [116,[x1=38,x2=9,x3=19,x4=3]]
```

Additional variable bounds and options may be provided alongside the file name. Note that the original constraints (those which are read from file) cannot be removed.
Input :

```
lpsolve("/path/to/file/somefile.lp",x2=1..8,x3=-10..10,
                lp_integervariables=[x4])
```

Output :

```
[82,[x1=38,x2=6,x3=10,x4=2]]
```

It is advisable to use only (capital) letters, digits and underscore when naming variables in a LP file, although the corresponding format allows many more characters. That is because these names are converted to Giac identifiers during the loading process.

Warning! Too large problems won't be loaded. More precisely, if $n_v \cdot n_c > 10^5$, where n_v is the number of variables and n_c is the number of constraints, loading is aborted. Many MPS files available, for example, in the Netlib repository (`http://www.netlib.org/`), contain very large problems with thousands of variables and constraints. Trying to load them to Xcas without a safety limit could easily eat up huge amounts of available memory, probably freezing up the whole system. If a large LP problem needs to be solved, one may consider using GLPK standalone solver[1].

5.45.3 Nonlinear optimization: `nlpsolve`

`nlpsolve` computes the optimum of a (not necessarily differentiable) nonlinear (multivariate) objective function, subject to a set of nonlinear equality and/or inequality constraints, using the COBYLA algorithm. The command takes the following arguments:

- `obj` : objective expression
- `constr` : list of equality and inequality constraints (optional)
- `bd` : sequence of variable boundaries (optional) : `x=a..b`, `y=c..d`, ...
- `opt` : sequence of options (optional), which may be one of:
  - `maximize=true` or `false` (or just `maximize`)
  - `nlp_initialpoint=[x=x0,y=y0,...]`
  - `nlp_iterationlimit=n`
  - `assume=nlp_nonnegative`
  - `nlp_precision=eps`

[1] See `https://www.gnu.org/software/glpk/` for installing GLPK in Linux or `http://winglpk.sourceforge.net/` for MS Windows.

nlpsolve returns a list containing the optimal value of the objective and a vector of optimal values of the decision variables.

The objective is minimized by default, unless maximize or maximize=true is specified as an option.

Initial point, if given, does not need to be feasible. Note, however, that the initial value of a variable must not be zero. If the initial point is not given or isn't feasible, a feasible starting guess is automatically generated. Note that choosing a good initial point is needed for obtaining a correct solution in some cases.

Input syntax for nlpsolve resembles that of Maple's NLPSolve (entering the objective as a function (univariate case) is not supported, however).

Examples

Input :

```
nlpsolve(ln(1+x1^2)-x2,[(1+x1^2)^2+x2^2=4])
```

Output :

```
[-1.73205080757,[x1=-4.77142305945e-08,x2=1.73205080757]]
```

Input :

```
nlpsolve(-x1*x2*x3,[72-x1-2x2-2x3>=0],
     x1=0..20,x2=0..11,x3=0..42)
```

Output :

```
[-3300.0,[x1=20.0,x2=11.0,x3=15.0]]
```

Input :

```
nlpsolve(x^3+2x*y-2y^2,x=-10..10,y=-10..10,
   nlp_initialpoint=[x=3,y=4],maximize)
```

Output :

```
[1050.0,[x=10.0,y=4.99999985519]]
```

Input :

```
nlpsolve(sin(x)/x,x=1..30)
```

Output :

```
[-0.217233628211,[x=4.49340942383]]
```

Input :

```
nlpsolve(2-1/120*x1*x2*x3*x4*x5,
[x1<=1,x2<=2,x3<=3,x4<=4,x5<=5],assume=nlp_nonnegative)
```

Output :

```
[1.0,[x1=1.0,x2=2.0,x3=3.0,x4=4.0,x5=5.0]]
```

5.45.4 Solving transportation problems: `tpsolve`

The objective of a transportation problem is to minimize the cost of distributing a product from m sources to n destinations. It is determined by three parameters :

- supply vector $\mathbf{s} = (s_1, s_2, \dots, s_m)$, where $s_k \in \mathbb{Z}$, $s_k > 0$ is the maximum number of units that can be delivered from k-th source for $k = 1, 2, \dots, m$,
- demand vector $\mathbf{d} = (d_1, d_2, \dots, d_n)$, where $d_k \in \mathbb{Z}$, $d_k > 0$ is the minimum number of units required by k-th destination for $k = 1, 2, \dots, n$,
- cost matrix $\mathbf{C} = [c_{ij}]_{m\times n}$, where $c_{ij} \in \mathbb{R}$, $c_{ij} \geq 0$ is the cost of transporting one unit of product from i-th source to j-th destination for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

The optimal solution is represented as matrix $\mathbf{X}^* = [x^*_{ij}]_{m\times n}$, where x^*_{ij} is number of units that must be transported from i-th source to j-th destination for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Function `tpsolve` accepts three arguments: supply vector, demand vector and cost matrix, respectively. It returns a sequence of two elements: the total (minimal) cost $c = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x^*_{ij}$ of transportation and the optimal solution $\mathbf{X}^*$.

Input :

```
s:=[12,17,11];d:=[10,10,10,10];
C:=[[50,75,30,45],[65,80,40,60],[40,70,50,55]];
tpsolve(s,d,C)
```

Output :

```
2020,[[0,0,2,10],[0,9,8,0],[10,1,0,0]]
```

If total supply and total demand are equal, i.e. if $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ holds, transportation problem is *closed* or *balanced*. If total supply exceeds total demand or vice versa, the problem is *unbalanced*. The excess supply/demand is covered by adding a dummy demand/supply point with zero cost of “transportation” from/to that point. Function `tpsolve` handles such cases automatically.

Input :

```
s:=[7,10,8,8,9,6];d:=[9,6,12,8,10];
C:=[[36,40,32,43,29],[28,27,29,40,38],[34,35,41,29,31],
[41,42,35,27,36],[25,28,40,34,38],[31,30,43,38,40]];
tpsolve(s,d,C)
```

Output :

```
1275,[[0,0,2,0,5],[0,0,10,0,0],[0,0,0,0,5],
[0,0,0,8,0],[9,0,0,0,0],[0,6,0,0,0]]
```

Sometimes it is desirable to forbid transportation on certain routes. That is usually achieved by setting very high cost to these routes, represented by symbol M. If `tpsolve` detects a symbol in the cost matrix, it interprets it as M and assigns 100 times larger cost than the largest numeric element of $\mathbf{C}$ to the corresponding routes, which forces the algorithm to avoid them.

Input :

```
s:=[95,70,165,165];d:=[195,150,30,45,75];
C:=[[15,M,45,M,0],[12,40,M,M,0],
[0,15,25,25,0],[M,0,M,12,0]]
tpsolve(s,d,C)
```

Output :

```
2820,[[20,0,0,0,75],[70,0,0,0,0],
[105,0,30,30,0],[0,150,0,15,0]]
```

5.46 Nonlinear optimization

5.46.1 Global extrema: `minimize maximize`

The function `minimize` takes four arguments :

- `obj` : univariate or multivariate expression
- `constr` (optional) : list of equality and inequality constraints
- `vars` : list of variables
- `location` (optional) : option keyword which may be `coordinates`, `locus` or `point`

The expression `obj` is minimized on the domain specified by constraints and/or bounding variables, which can be done as specifying e.g. `x=a..b` in `vars`. The domain must be closed and bounded and `obj` must be continuous in every point of it. Else, the final result may be incorrect or meaningless.

Constraints may be given as equalities or inequalities, but also as expressions which are assumed to be equal to zero. If there is only one constraint, the list delimiters may be dropped. The same applies to the specification of variables.

`minimize` returns minimal value. If it could not be obtained, it returns `undef`. If `location` is specified, the list of points where the minimum is achieved is also returned as the second member in a sequence. Keywords `locus`, `coordinates` and `point` all have the same effect.

The function `maximize` takes the same parameters as `minimize`. The difference is that it computes global maximum of `obj` on the specified domain.

Examples

Input :

```
minimize(sin(x),[x=0..4])
```

Output :

```
sin(4)
```

Input :

```
minimize(asin(x),x=-1..1)
```

Output :

```
-pi/2
```

Input :

```
minimize(x^4-x^2,x=-3..3,locus)
```

Output :

```
-1/4,[-sqrt(2)/2]
```

Input :

```
minimize(x-abs(x),x=-1..1)
```

Output :

```
-2
```

Input :

```
minimize(when(x==0,0,exp(-1/x^2)),x=-1..1)
```

Output :

```
0
```

Input :

```
minimize(sin(x)+cos(x),x=0..20,coordinates)
```

Output :

```
-sqrt(2),[5*pi/4,13*pi/4,21*pi/4]
```

Input :

```
minimize(x^2-3x+y^2+3y+3,[x=2..4,y=-4..-2],point)
```

Output :

```
-1,[[2,-2]]
```

Input :

```
obj:=sqrt(x^2+y^2)-z;
constr:=[x^2+y^2<=16,x+y+z=10];
minimize(obj,constr,[x,y,z])
```

Output :

```
-4*sqrt(2)-6
```

Input :

```
minimize(x^2*(y+1)-2y,[y<=2,sqrt(1+x^2)<=y],[x,y])
```

Output :

```
-4
```

Input :

```
maximize(cos(x),x=1..3)
```

Output :

```
cos(1)
```

Input :

```
obj:=piecewise(x<=-2,x+6,x<=1,x^2,3/2-x/2);
maximize(obj,x=-3..2)
```

Output :

```
4
```

Input :

```
maximize(x*y*z,x^2+2*y^2+3*z^2<=1,[x,y,z])
```

Output :

```
sqrt(2)/18
```

Input :

```
maximize(x*y,[x+y^2<=2,x>=0,y>=0],[x,y],locus)
```

Output :

```
4*sqrt(6)/9,[[4/3,sqrt(6)/3]]
```

Input :

```
maximize(y^2-x^2*y,y<=x,[x=0..2,y=0..2])
```

Output :

```
4/27
```

Input :

```
assume(a>0);
maximize(x^2*y^2*z^2,x^2+y^2+z^2=a^2,[x,y,z])
```

Output :

```
a^6/27
```

5.46.2 Local extrema: `extrema`

Local extrema of a univariate or multivariate differentiable function under equality constraints can be obtained by using function `extrema` which takes four arguments :

- `expr` : differentiable expression
- `constr` (optional) : list of equality constraints
- `vars` : list of variables
- `order_size=<positive integer>` (optional) : upper bound for the order of derivatives examined in the process (defaults to 5)

Function returns sequence of two lists of points: local minima and maxima, respectively. Saddle and unclassified points are reported in the message area. Also, information about possible (non)strict extrema is printed out.

A single constraint/variable can be specified without list delimiters. A constraint may be specified as an equality or expression which is assumed to be equal to zero.

Number of constraints must be strictly less than number of variables. Additionally, denoting k-th constraint by $g_k(x_1, x_2, \ldots, x_n) = 0$ for $k = 1, 2, \ldots, m$ and letting $\mathbf{g} = (g_1, g_2, \ldots, g_m)$, Jacobian matrix of $\mathbf{g}$ has to be full rank (i.e. equal to m), since implicit differentiation is performed.

Variables may be specified with bounds, e.g. `x=a..b`, which is interpreted as $x \in (a, b)$. For semi-bounded variables one can use `-infinity` for a or `+infinity` for b. Also, parameter `vars` may be entered as e.g. `[x1=a1,x2=a2,...,xn=an]`, in which case the critical point close to $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is computed numerically, applying an iterative method with initial point $\mathbf{a}$.

If `order_size=<n>` is specified as the fourth argument, derivatives up to order n are inspected to find critical points and classify them. For `order_size=1` the function returns a single list containing all critical points found. The default is $n = 5$. If some critical points are left unclassified one might consider repeating the process with larger value of n, although the success is not guaranteed.

Examples

Input :

```
extrema(-2*cos(x)-cos(x)^2,x)
```

Output :

```
[0],[pi]
```

Input :

```
extrema(x/2-2*sin(x/2),x=-12..12)
```

Output :

```
[2*pi/3,-10*pi/3],[10*pi/3,-2*pi/3]
```

Input :

```
assume(a>=0);extrema(x^2+a*x,x)
```

Output :

```
[-a/2],[]
```

Input :

```
extrema(exp(x^2-2x)*ln(x)*ln(1-x),x=0.5)
```

Output :

```
[],[0.277769149124]
```

Input :

```
extrema(x^3-2x*y+3y^4,[x,y])
```

Output :

```
[[12^(1/5)/3,(12^(1/5))^2/6]],[]
```

Input :

```
assume(a>0);extrema(x/a^2+a*y^2,x+y=a,[x,y])
```

Output :

```
[[(2*a^4-1)/(2*a^3),1/(2*a^3)]],[]
```

Input :

```
extrema(x^2+y^2,x*y=1,[x=0..inf,y=0..inf])
```

Output :

```
[[1,1]],[]
```

Input :

```
extrema(x2^4-x1^4-x2^8+x1^10,[x1,x2])
```

Output :

```
[[6250^(1/6)/5,0],[-6250^(1/6)/5,0]],[]
```

Input :

```
extrema(x*y*z,x+y+z=1,[x,y,z],order_size=1)
```

Output :

```
[[1,0,0],[0,1,0],[0,0,1],[1/3,1/3,1/3]]
```

5.46.3 Minimax polynomial approximation: `minimax`

The function `minimax` is called by entering :

```
minimax(expr,var=a..b,n,[limit=m])
```

where `expr` is an univariate expression (e.g. $f(x)$) to approximate, `var` is a variable (e.g. x), $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$. Expression `expr` must be continuous on $[a, b]$. The function returns minimax polynomial (e.g. $p(x)$) of degree n or lower that approximates `expr` on $[a, b]$. The approximation is found by applying Remez algorithm.

If the fourth argument is specified, m is used to limit the number of iterations of the algorithm. It is unlimited by default.

The largest absolute error of the approximation $p(x)$, i.e. $\max_{a\leq x\leq b} |f(x) - p(x)|$, is printed in the message area.

Since the coefficients of p are computed numerically, one should avoid setting n unnecessary high as it may result in a poor approximation due to the roundoff errors.

Input :

```
minimax(sin(x),x=0..2*pi,10)
```

Output :

```
5.8514210172e-06+0.999777263385*x+0.00140015265723*x^2
-0.170089663733*x^3+0.0042684304696*x^4+
0.00525794766407*x^5+0.00135760214958*x^6
-0.000570502074548*x^7+6.07297119422e-05*x^8
-2.14787414001e-06*x^9-2.97767481643e-15*x^10
```

The largest absolute error of this approximation is $5.85234008632 \times 10^{-6}$.

5.47 Different matrix norm

5.47.1 l^2 matrix norm : `norm l2norm`

`norm` (or `l2norm`) takes as argument a matrix $A = a_{j,k}$ (see also 5.39.1).

`norm` (or `l2norm`) returns $\sqrt{\sum_{j,k} a_{j,k}^2}$.

Input :

```
norm([[1,2],[3,-4]])
```

or :

```
l2norm([[1,2],[3,-4]])
```

Output :

```
sqrt(30)
```

5.47.2 l^∞ matrix norm : `maxnorm`

`maxnorm` takes as argument a matrix $A = a_{j,k}$ (see also 5.39.1).
`maxnorm` returns $\max(|a_{j,k}|)$.
Input :

```
maxnorm([[1,2],[3,-4]])
```

Output :

```
4
```

5.47.3 Matrix row norm : `rownorm rowNorm`

`rownorm` (or `rowNorm`) takes as argument a matrix $A = a_{j,k}$.
`rownorm` (or `rowNorm`) returns $\max_k(\sum_j |a_{j,k}|)$.
Input :

```
rownorm([[1,2],[3,-4]])
```

or :

```
rowNorm([[1,2],[3,-4]])
```

Output :

```
7
```

Indeed : $\max(1 + 2, 3 + 4) = 7$

5.47.4 Matrix column norm : `colnorm colNorm`

`colnorm` (or `colNorm`) takes as argument a matrix $A = a_{j,k}$.
`colnorm` (or `colNorm`) returns $\max_j(\sum_k(|a_{j,k}|))$.
Input :

```
colnorm([[1,2],[3,-4]])
```

or :

```
colNorm([[1,2],[3,-4]])
```

Output :

```
6
```

Indeed : $\max(1 + 3, 2 + 4) = 6$

5.48 Matrix reduction

5.48.1 Eigenvalues : eigenvals

`eigenvals` takes as argument a square matrix A of size n.
`eigenvals` returns the sequence of the n eigenvalues of A.
Remark : If A is exact, `Xcas` may not be able to find the exact roots of the characteristic polynomial, `eigenvals` will return approximate eigenvalues of A if the coefficients are numeric or a subset of the eigenvalues if the coefficients are symbolic.
Input :

```
eigenvals([[4,1,-2],[1,2,-1],[2,1,0]])
```

Output :

```
(2,2,2)
```

Input :

```
eigenvals([[4,1,0],[1,2,-1],[2,1,0]])
```

Output :

```
(0.324869129433,4.21431974338,1.46081112719)
```

5.48.2 Eigenvalues : egvl eigenvalues eigVl

`egvl` (or `eigenvalues eigVl`) takes as argument a square matrix A of size n.
`egvl` (or `eigenvalues eigVl`) returns the Jordan normal form of A.
Remark : If A is exact, `Xcas` may not be able to find the exact roots of the characteristic polynomial, `eigenvalues` will return an approximate diagonalization of A if the coefficients are numeric.
Input :

```
egvl([[4,1,-2],[1,2,-1],[2,1,0]])
```

Output :

```
[[2,1,0],[0,2,1],[0,0,2]]
```

Input :

```
egvl([[4,1,0],[1,2,-1],[2,1,0]])
```

Output :

```
[[0.324869129433,0,0],[0,4.21431974338,0],[0,0,1.46081112719]]
```

5.48.3 Eigenvectors : egv eigenvectors eigenvects eigVc

egv (or eigenvectors eigenvects eigVc) takes as argument a square matrix A of size n.
If A is a diagonalizable matrix, egv (or eigenvectors eigenvects eigVc) returns a matrix whose columns are the eigenvectors of the matrix A. Otherwise, it will fail (see also jordan for characteristic vectors).
Input :

```
egv([[1,1,3],[1,3,1],[3,1,1]])
```

Output :

```
[[-1,1,1],[2,1,0],[-1,1,-1]]
```

Input :

```
egv([[4,1,-2],[1,2,-1],[2,1,0]])
```

Output :

```
"Not diagonalizable at eigenvalue 2"
```

In complex mode, input :

```
egv([[2,0,0],[0,2,-1],[2,1,2]])
```

Output :

```
[0,1,0],[-1,-2,-1],[i,0,-i]]
```

5.48.4 Rational Jordan matrix : rat_jordan

rat_jordan takes as argument a square matrix A of size n with exact coefficients.
rat_jordan returns :

- in Xcas, Mupad or TI mode
  a sequence of two matrices : a matrix P (the columns of P are the eigenvectors if A is diagonalizable in the field of its coefficients) and the rational Jordan matrix J of A, that is the most reduced matrix in the field of the coefficients of A (or the complexified field in complex mode), where
  $$J = P^{-1}AP$$
- in Maple mode
  the Jordan matrix J of A. We can also have the matrix P verifying $J = P^{-1}AP$ in a variable by passing this variable as second argument, for example

```
rat_jordan([[1,0,0],[1,2,-1],[0,0,1]],'P')
```

Remarks

- the syntax `Maple` is also valid in the other modes, for example, in `Xcas` mode input

  ```
  rat_jordan([[4,1,1],[1,4,1],[1,1,4]],'P')
  ```

  Output :

  ```
  [[1,-1,1/2],[1,0,-1],[1,1,1/2]]
  ```

  then `P` returns

  ```
  [[6,0,0],[0,3,0],[0,0,3]]
  ```

- the coefficients of P and J belongs to the same field as the coefficients of A. For example, in `Xcas` mode, input :

  ```
  rat_jordan([[1,0,1],[0,2,-1],[1,-1,1]])
  ```

  Output :

  ```
  [[1,1,2],[0,0,-1],[0,1,2]],[[0,0,-1],[1,0,-3],[0,1,4]]
  ```

  Input (put `-pcar(...)` because the argument of `companion` is a unit polynomial (see 5.48.10)

  ```
  companion(-pcar([[1,0,1],[0,2,-1],[1,-1,1]],x),x)
  ```

  Output :

  ```
  [[0,0,-1],[1,0,-3],[0,1,4]]
  ```

  Input :

  ```
  rat_jordan([[1,0,0],[0,1,1],[1,1,-1]])
  ```

  Output :

  ```
  [[-1,0,0],[1,1,1],[0,0,1]],[[1,0,0],[0,0,2],[0,1,0]]
  ```

  Input :

  ```
  factor(pcar([[1,0,0],[0,1,1],[1,1,-1]],x))
  ```

  Output :

  ```
  -(x-1)*(x^2-2)
  ```

  Input :

```
companion((x^2-2),x)
```

Output :

```
[[0,2],[1,0]]
```

- When A is symmetric and has eigenvalues with an multiple order, `Xcas` returns orthogonal eigenvectors (not always of norm equal to 1) i.e. `tran(P)*P` is a diagonal matrix where the diagonal is the square norm of the eigenvectors, for example :

  ```
  rat_jordan([[4,1,1],[1,4,1],[1,1,4]])
  ```

  returns :

  ```
  [[1,-1,1/2],[1,0,-1],[1,1,1/2]],[[6,0,0],[0,3,0],[0,0,3]]
  ```

Input in `Xcas`, `Mupad` or `TI` mode :

```
rat_jordan([[1,0,0],[1,2,-1],[0,0,1]])
```

Output :

```
[[0,1,0],[1,0,1],[0,1,1]],[[2,0,0],[0,1,0],[0,0,1]]
```

Input in `Xcas`, `Mupad` or `TI` mode :

```
rat_jordan([[4,1,-2],[1,2,-1],[2,1,0]])
```

Output :

```
[[[1,2,1],[0,1,0],[1,2,0]],[[2,1,0],[0,2,1],[0,0,2]]]
```

In complex mode and in `Xcas`, `Mupad` or `TI` mode , input :

```
rat_jordan([[2,0,0],[0,2,-1],[2,1,2]])
```

Output :

```
[[1,0,0],[-2,-1,-1],[0,-i,i]],[[2,0,0],[0,2-i,0],[0,0,2+i]]
```

Input in `Maple` mode :

```
rat_jordan([[1,0,0],[1,2,-1],[0,0,1]],'P')
```

Output :

```
[[2,0,0],[0,1,0],[0,0,1]]
```

then input :

```
P
```

Output :

```
[[0,1,0],[1,0,1],[0,1,1]]]
```

5.48.5 Jordan normal form : `jordan`

`jordan` takes as argument a square matrix A of size n.
`jordan` returns :

- in `Xcas`, `Mupad` or `TI` mode
  a sequence of two matrices : a matrix P whose columns are the eigenvectors and characteristic vectors of the matrix A and the Jordan matrix J of A verifying $J = P^{-1}AP$,
- in `Maple` mode
  the Jordan matrix J of A. We can also have the matrix P verifying $J = P^{-1}AP$ in a variable by passing this variable as second argument, for example

  ```
  jordan([[1,0,0],[0,1,1],[1,1,-1]],'P')
  ```

Remarks

- the `Maple` syntax is also valid in the other modes, for example, in `Xcas` mode input :

  ```
  jordan([[4,1,1],[1,4,1],[1,1,4]],'P')
  ```

  Output :

  ```
  [[1,-1,1/2],[1,0,-1],[1,1,1/2]]
  ```

  then `P` returns

  ```
  [[6,0,0],[0,3,0],[0,0,3]]
  ```

- When A is symmetric and has eigenvalues with multiple orders, `Xcas` returns orthogonal eigenvectors (not always of norm equal to 1) i.e. `tran(P)*P` is a diagonal matrix where the diagonal is the square norm of the eigenvectors, for example :

  ```
  jordan([[4,1,1],[1,4,1],[1,1,4]])
  ```

  returns :

  ```
  [[1,-1,1/2],[1,0,-1],[1,1,1/2]],[[6,0,0],[0,3,0],[0,0,3]]
  ```

Input in `Xcas`, `Mupad` or `TI` mode :

```
jordan([[1,0,0],[0,1,1],[1,1,-1]])
```

Output :

```
[[1,0,0],[0,1,1],[1,1,-1]],[[-1,0,0],[1,1,1],[0,-sqrt(2)-1,sqrt(2)-1]
```

Input in `Maple` mode :

```
jordan([[1,0,0],[0,1,1],[1,1,-1]])
```

Output :

```
[[1,0,0],[0,-(sqrt(2)),0],[0,0,sqrt(2)]]
```

then input :

```
P
```

Output :

```
[[-1,0,0],[1,1,1],[0,-sqrt(2)-1,sqrt(2)-1]]
```

Input in `Xcas`, `Mupad` or `TI` mode :

```
jordan([[4,1,-2],[1,2,-1],[2,1,0]])
```

Output :

```
[[[1,2,1],[0,1,0],[1,2,0]],[[2,1,0],[0,2,1],[0,0,2]]]
```

In complex mode and in `Xcas`, `Mupad` or `TI` mode , input :

```
jordan([[2,0,0],[0,2,-1],[2,1,2]])
```

Output :

```
[[1,0,0],[-2,-1,-1],[0,-i,i]],[[2,0,0],[0,2-i,0],[0,0,2+i]]
```

5.48.6 Characteristic polynomial : `charpoly`

`charpoly` (or `pcar`) takes one or two argument(s), a square matrix A of size n and optionally the name of a symbolic variable.
`charpoly` returns the characteristic polynomial P of A written as the list of its coefficients if no variable name was provided or written as an expression with respect to the variable name provided as second argument.
The characteristic polynomial P of A is defined as

$$P(x) = \det(xI - A)$$

Input :

```
charpoly([[4,1,-2],[1,2,-1],[2,1,0]])
```

Output :

```
[1,-6,12,-8]
```

Hence, the characteristic polynomial of this matrix is $x^3 - 6x^2 + 12x - 8$ (input `normal(poly2symb([1,-6,12,-8],x))` to get its symbolic representation).
Input :

```
purge(X):; charpoly([[4,1,-2],[1,2,-1],[2,1,0]],X)
```

Output :

```
X^3-6*X^2+12*X-8
```

5.48.7 Characteristic polynomial using Hessenberg algorithm : pcar_hessenberg

pcar_hessenberg takes as argument a square matrix A of size n and optionally the name of a symbolic variable.
pcar_hessenberg returns the characteristic polynomial P of A written as the list of its coefficients if no variable was provided or written in its symbolic form with respect to the variable name given as second argument, where

$$P(x) = \det(xI - A)$$

The characteristic polynomial is computed using the Hessenberg algorithm (see e.g. Cohen) which is more efficient ($O(n^3)$ deterministic) if the coefficients of A are in a finite field or use a finite representation like approximate numeric coefficients. Note however that this algorithm behaves badly if the coefficients are e.g. in $\mathbb{Q}$.
Input :

```
pcar_hessenberg([[4,1,-2],[1,2,-1],[2,1,0]] % 37)
```

Output :

```
[1 % 37 ,-6% 37,12 % 37,-8 % 37]
```

Input :

```
pcar_hessenberg([[4,1,-2],[1,2,-1],[2,1,0]] % 37,x)
```

Output :

```
x^3-6 %37 *x^2+12 % 37 *x-8 % 37
```

Hence, the characteristic polynomial of [[4,1,-2],[1,2,-1],[2,1,0]] in $\mathbb{Z}/37\mathbb{Z}$ is

$$x^3 - 6x^2 + 12x - 8$$

5.48.8 Minimal polynomial : pmin

pmin takes one (resp. two) argument(s): a square matrix A of size n and optionally the name of a symbolic variable.
pmin returns the minimal polynomial of A written as a list of its coefficients if no variable was provided, or written in symbolic form with respect to the variable name given as second argument. The minimal polynomial of A is the polynomial P having minimal degree such that $P(A) = 0$.
Input :

```
pmin([[1,0],[0,1]])
```

Output :

```
[1,-1]
```

Input :

```
pmin([[1,0],[0,1]],x)
```

Output :

```
x-1
```

Hence the minimal polynomial of [[1,0],[0,1]] is `x-1`.
Input :

```
pmin([[2,1,0],[0,2,0],[0,0,2]])
```

Output :

```
[1,-4,4]
```

Input :

```
pmin([[2,1,0],[0,2,0],[0,0,2]],x)
```

Output :

```
x^2-4*x+4
```

Hence, the minimal polynomial of [[2,1,0],[0,2,0],[0,0,2]] is $x^2 - 4x + 4$.

5.48.9 Adjoint matrix : `adjoint_matrix`

`adjoint_matrix` takes as argument a square matrix A of size n.
`adjoint_matrix` returns the list of the coefficients of P (the characteristic polynomial of A), and the list of the matrix coefficients of Q (the adjoint matrix of A).

The comatrix of a square matrix A of size n is the matrix B defined by $A \times B = \det(A) \times I$. The adjoint matrix of A is the comatrix of $xI - A$. It is a polynomial of degree $n - 1$ in x having matrix coefficients. The following relation holds:

$$P(x) \times I = \det(xI - A)I = (xI - A)Q(x)$$

Since the polynomial $P(x) \times I - P(A)$ (with matrix coefficients) is also divisible by $x \times I - A$ (by algebraic identities), this proves that $P(A) = 0$. We also have $Q(x) = I \times x^{n-1} + ... + B_0$ where $B_0 =$ is the comatrix of A (up to the sign if n is odd).
Input :

```
adjoint_matrix([[4,1,-2],[1,2,-1],[2,1,0]])
```

Output :

```
[ [1,-6,12,-8],
[ [[1,0,0],[0,1,0],[0,0,1]], [[-2,1,-2],
[1,-4,-1],[2,1,-6]], [[1,-2,3],[-2,4,2],[-3,-2,7]] ] ]
```

Hence the characteristic polynomial is :

$$P(x) = x^3 - 6 * x^2 + 12 * x - 8$$

The determinant of A is equal to $-P(0) = 8$. The comatrix of A is equal to :

$$B = Q(0) = [[1, -2, 3], [-2, 4, 2], [-3, -2, 7]]$$

Hence the inverse of A is equal to :

$$1/8 * [[1, -2, 3], [-2, 4, 2], [-3, -2, 7]]$$

The adjoint matrix of A is :

$[[x^2-2x+1, x-2, -2x+3], [x-2, x^2-4x+4, -x+2], [2x-3, x-2, x^2-6x+7]]$

Input :

```
adjoint_matrix([[4,1],[1,2]])
```

Output :

```
[[1,-6,7],[[[1,0],[0,1]],[[-2,1],[1,-4]]]]
```

Hence the characteristic polynomial P is :

$$P(x) = x^2 - 6 * x + 7$$

The determinant of A is equal to $+P(0) = 7$. The comatrix of A is equal to

$$Q(0) = -[[-2, 1], [1, -4]]$$

Hence the inverse of A is equal to :

$$-1/7 * [[-2, 1], [1, -4]]$$

The adjoint matrix of A is :

$$-[[x - 2, 1], [1, x - 4]]$$

5.48.10 Companion matrix of a polynomial : `companion`

`companion` takes as argument an unitary polynomial P and the name of its variable.
`companion` returns the matrix whose characteristic polynomial is P.
If $P(x) = x^n+a_{n-1}x^{n-1}+...+a_{-1}x+a_0$, this matrix is equal to the unit matrix of size $n-1$ bordered with $[0, 0.., 0, -a_0]$ as first row, and with $[-a_0, -a_1, ..., -a_{n-1}]$ as last column.
Input :

```
companion(x^2+5x-7,x)
```

Output :

```
[[0,7],[1,-5]]
```

Input :

```
companion(x^4+3x^3+2x^2+4x-1,x)
```

Output :

```
[[0,0,0,1],[1,0,0,-4],[0,1,0,-2],[0,0,1,-3]]
```

5.48.11 Hessenberg matrix reduction : `hessenberg`

`hessenberg` takes as argument a matrix A.
`hessenberg` returns a matrix B equivalent to A where the coefficients below the sub-principal diagonal are zero. B is a Hessenberg matrix.
Input :

```
hessenberg([[3,2,2,2,2],[2,1,2,-1,-1],[2,2,1,-1,1],
          [2,-1,-1,3,1],[2,-1,1,1,2]])
```

Output :

```
[[3,8,5,10,2],[2,1,1/2,-5,-1],[0,2,1,8,2],
     [0,0,1/2,8,1],[0,0,0,-26,-3]]
```

Input

```
A:=[[3,2,2,2,2],[2,1,2,-1,-1],[2,2,1,-1,1],
     [2,-1,-1,3,1],[2,-1,1,1,2]] :;
  B:= hessenberg(A):; pcar(A); pcar(B)
```

Output: `[1,-7,-66,-24]`.

5.48.12 Hermite normal form : `ihermite`

`ihermite` takes as argument a matrix `A` with coefficients in $\mathbb{Z}$.
`ihermite` returns two matrices `U` and `B` such that `B=U*A`, `U` is invertible in $\mathbb{Z}$ ($\det(U) = \pm 1$) and `B` is upper-triangular. Moreover, the absolute value of the coefficients above the diagonal of `B` are smaller than the pivot of the column divided by 2.

The answer is obtained by a Gauss-like reduction algorithm using only operations of rows with integer coefficients and invertible in $\mathbb{Z}$.
Input :

```
A:=[[9,-36,30],[-36,192,-180],[30,-180,180]];
            U,B:=ihermite(A)
```

Output :

```
 [[9,-36,30],[-36,192,-180],[30,-180,180]],
[[13,9,7],[6,4,3],[20,15,12]],[[3,0,30],[0,12,0],[0,0,60]]
```

Application: Compute a $\mathbb{Z}$-basis of the kernel of a matrix having integer coefficients
Let `M` be a matrix with integer coefficients.
Input :

```
(U,A):=ihermite(transpose(M)).
```

This returns U and A such that `A=U*transpose(M)` hence
`transpose(A)=M*transpose(U)`.
The columns of `transpose(A)` which are identically 0 (at the right, coming from the rows of A which are identically 0 at the bottom) correspond to columns

of transpose(U) which form a basis of Ker(M). In other words, the rows of A which are identically 0 correspond to rows of U which form a basis of Ker(M).
Example
Let M:=[[1,4,7],[2,5,8],[3,6,9]]. Input

```
U,A:=ihermite(tran(M))
```

Output

```
U:=[[-3,1,0],[4,-1,0],[-1,2,-1]] and
A:=[[1,-1,-3],[0,3,6],[0,0,0]]
```

Since A[2]=[0,0,0], a ℤ-basis of Ker(M) is U[2]=[-1,2,-1].
Verification M*U[2]=[0,0,0].

5.48.13 Smith normal form : ismith

ismith takes as argument a matrix with coefficients in ℤ.
ismith returns three matrices U,B and V such that B=U*A*V, U and V are invertible in ℤ, B is diagonal, and B[i,i] divides B[i+1,i+1]. The coefficients B[i,i] are called invariant factors, they are used to describe the structure of finite abelian groups.
Input :

```
A:=[[9,-36,30],[-36,192,-180],[30,-180,180]];
U,B,V:=ismith(A)
```

Output :

```
[[-3,0,1],[6,4,3],[20,15,12]],
[[3,0,0],[0,12,0],[0,0,60]],
[[1,24,-30],[0,1,0],[0,0,1]]
```

The invariant factors are 3, 12 and 60.

5.49 Isometries

5.49.1 Recognize an isometry : isom

isom takes as argument the matrix of a linear function in dimension 2 or 3.
isom returns :

- if the linear function is a direct isometry,
  the list of the characteristic elements of this isometry and +1,
- if the linear function is an indirect isometry,
  the list of the characteristic elements of this isometry and -1
- if the linear function is not an isometry,
  [0].

Input :

```
isom([[0,0,1],[0,1,0],[1,0,0]])
```

Output :

```
[[1,0,-1],-1]
```

which means that this isometry is a 3-d symmetry with respect to the plane $x - z = 0$.
Input :

```
isom(sqrt(2)/2*[[1,-1],[1,1]])
```

Output :

```
[pi/4,1]
```

Hence, this isometry is a 2-d rotation of angle $\frac{\pi}{4}$.
Input :

```
isom([[0,0,1],[0,1,0],[0,0,1]])
```

Output :

```
[0]
```

therefore this transformation is not an isometry.

5.49.2 Find the matrix of an isometry : mkisom

mkisom takes as argument :

- In dimension 3, the list of characteristic elements (axis direction, angle for a rotation or normal to the plane for a symmetry) and +1 for a direct isometry or -1 an indirect isometry.
- In dimension 2, a characteristic element (an angle or a vector) and +1 for a direct isometry (rotation) or -1 for an indirect isometry (symmetry).

mkisom returns the matrix of the corresponding isometry.
Input :

```
mkisom([[-1,2,-1],pi],1)
```

Output the matrix of the rotation of axis $[-1, 2, -1]$ and angle π:

```
[[-2/3,-2/3,1/3],[-2/3,1/3,-2/3],[1/3,-2/3,-2/3]]
```

Input :

```
mkisom([pi],-1)
```

Output the matrix of the symmetry with respect to O :

```
[[-1,0,0],[0,-1,0],[0,0,-1]]
```

Input :

```
mkisom([1,1,1],-1)
```

Output the matrix of the symmetry with respect to the plane $x + y + z = 0$:

```
[[1/3,-2/3,-2/3],[-2/3,1/3,-2/3],[-2/3,-2/3,1/3]]
```

Input :

```
mkisom([[1,1,1],pi/3],-1)
```

Output the matrix of the product of a rotation of axis $[1, 1, 1]$ and angle $\frac{\pi}{3}$ and of a symmetry with respect to the plane $x + y + z = 0$:

```
[[0,-1,0],[0,0,-1],[-1,0,0]]
```

Input :

```
mkisom(pi/2,1)
```

Output the matrix of the plane rotation of angle $\frac{\pi}{2}$:

```
[[0,-1],[1,0]]
```

Input :

```
mkisom([1,2],-1)
```

Output matrix of the plane symmetry with respect to the line of equation $x + 2y = 0$:

```
[[3/5,-4/5],[-4/5,-3/5]]
```

5.50 Matrix factorizations

Note that most matrix factorization algorithms are implemented numerically, only a few of them will work symbolically.

5.50.1 Cholesky decomposition : cholesky

`cholesky` takes as argument a square symmetric positive definite matrix `M` of size n.
`cholesky` returns a symbolic or numeric matrix `P`. `P` is a lower triangular matrix such that :

```
tran(P)*P=M
```

Input :

```
cholesky([[1,1],[1,5]])
```

Output :

```
[[1,0],[1,2]]
```

Input :

```
cholesky([[3,1],[1,4]])
```

Output :

```
[[sqrt(3),0],[(sqrt(3))/3,(sqrt(33))/3]]
```

Input :

```
cholesky([[1,1],[1,4]])
```

Output :

```
[[1,0],[1,sqrt(3)]]
```

Warning If the matrix argument A is not a symmetric matrix, `cholesky` does not return an error, instead `cholesky` will use the symmetric matrix B of the the quadratic form q corresponding to the (non symmetric) bilinear form of the matrix A.
Input :

```
cholesky([[1,-1],[-1,4]])
```

or :

```
cholesky([[1,-3],[1,4]])
```

Output :

```
[[1,0],[-1,sqrt(3)]]
```

5.50.2 QR decomposition : `qr`

`qr` takes as argument a numeric square matrix A of size n.
`qr` factorizes numerically this matrix as $Q * R$ where Q is an orthogonal matrix ($^tQ * Q = I$) and R is an upper triangular matrix. `qr(A)` returns only `R`, run `Q=A*inv(R)` to get `Q`.
Input :

```
qr([[3,5],[4,5]])
```

Output is the matrix `R` :

```
[[-5,-7],[0,-1]]
```

Input :

```
qr([[1,2],[3,4]])
```

Output is the matrix `R` :

```
[[-3.16227766017,-4.42718872424],[0,-0.632455532034]]
```

5.50.3 QR decomposition (for TI compatibility) : `QR`

`QR` takes as argument a numeric square matrix A of size n and two variable names, `var1` and `var2`.
`QR` factorizes this matrix numerically as $Q * R$ where Q is an orthogonal matrix (${}^tQ * Q = I$) and R is an upper triangular matrix. `QR(A,var1,var2)` returns `R`, stores `Q=A*inv(R)` in `var1` and `R` in `var2`.
Input :

```
QR([[3,5],[4,5]],Q,R)
```

Output the matrix `R` :

```
[[-5,-7],[0,-1]]
```

Then input :

```
Q
```

Output the matrix `Q` :

```
[[-0.6,-0.8],[-0.8,0.6]]
```

5.50.4 LU decomposition : `lu`

`lu` takes as argument a square matrix A of size n (numeric or symbolic).
`lu(A)` returns a permutation p of $0..n-1$, a lower triangular matrix L, with 1s on the diagonal, and an upper triangular matrix U, such that :

- $P * A = L * U$ where P is the permutation matrix associated to p (that may be computed by `P:=permu2mat(p)`),
- the equation $A * x = B$ is equivalent to :

  $L*U*x = P*B = p(B)$ where $p(B) = [b_{p(0)}, b_{p(1)}..b_{p(n-1)}], \quad B = [b_0, b_1..b_{n-1}]$

The permutation matrix P is defined from p by :

$$P[i, p(i)] = 1, \quad P[i, j] = 0 \text{ if } j \neq p(i)$$

In other words, it is the identity matrix where the rows are permuted according to the permutation p. The function `permu2mat` may be used to compute P (`permu2mat(p)` returns `P`).
Input :

```
(p,L,U):=lu([[3.,5.],[4.,5.]])
```

Output :

```
[1,0],[[1,0],[0.75,1]],[[4,5],[0,1.25]]
```

Here $n = 2$, hence :

$$P[0, p(0)] = P_2[0, 1] = 1, \quad P[1, p(1)] = P_2[1, 0] = 1, \quad P = [[0, 1], [1, 0]]$$

Verification :
Input :

```
permu2mat(p)*A; L*U
```

Output:

```
[[4.0,5.0],[3.0,5.0]],[[4.0,5.0],[3.0,5.0]]
```

Note that the permutation is different for exact input (the choice of pivot is the simplest instead of the largest in absolute value).
Input :

```
lu([[1,2],[3,4]])
```

Output :

```
[1,0],[[1,0],[3,1]],[[1,2],[0,-2]]
```

Input :

```
lu([[1.0,2],[3,4]])
```

Output :

```
[1,0],[[1,0],[0.333333333333,1]],[[3,4],
        [0,0.666666666667]]
```

5.50.5 LU decomposition (for TI compatibility) : LU

LU takes as argument a numeric square matrix A of size n and three variable names, var1, var2 and var3.
LU(A,var1,var2,var3) returns P, a permutation matrix, and stores :

- a lower triangular matrix L, with 1 on the diagonal, in var1,
- an upper triangular matrix U in var2,
- the permutation matrix P, result of the command LU, in var3.

These matrices are such that

the equation $A * x = B$ is equivalent to $L * U * x = P * B$.

Input :

```
LU([[3,5],[4,5]],L,U,P)
```

Output :

```
[[0,1],[1,0]]
```

Input :

```
L
```

Output :

```
[[1,0],[0.75,1]]
```

Input :

```
U
```

Output :

```
[[4,5],[0,1.25]]
```

Input :

```
P
```

Output :

```
[[0,1],[1,0]]
```

5.50.6 Singular value decomposition : svd

svd (singular value decomposition) takes as argument a numeric square matrix of size n.
svd(A) returns an orthogonal matrix U, the diagonal s of a diagonal matrix S and an orthogonal matrix Q (${}^tQ * Q = I$) such that :

$$A = US^tQ$$

Input :

```
svd([[1,2],[3,4]])
```

Output :

```
[[-0.404553584834,-0.914514295677],[-0.914514295677,
 0.404553584834]], [5.46498570422,0.365966190626],
 [[-0.576048436766,0.81741556047],[-0.81741556047,
 -0.576048436766]]
```

Input :

```
(U,s,Q):=svd([[3,5],[4,5]])
```

Output :

```
[[-0.672988041811,-0.739653361771],[-0.739653361771,
 0.672988041811]],[8.6409011028,0.578643354497],
 [[-0.576048436766,0.81741556047],[-0.81741556047,
 -0.576048436766]]
```

Verification :
Input :

```
U*diag(s)*tran(Q)
```

Output :

```
[[3.0,5.0],[4.0,5.0]]
```

5.50.7 Short basis of a lattice : lll

lll takes as argument an invertible matrix M with integer coefficients.
lll returns (S, A, L, O) such that:

- the rows of S is a short basis of the $\mathbb{Z}$-module generated by the rows of M,
- A is the change-of-basis matrix from the short basis to the basis defined by the rows of M ($A * M = S$),
- L is a lower triangular matrix, the modulus of its non diagonal coefficients are less than 1/2,
- O is a matrix with orthogonal rows such that $L * O = S$.

Input :

```
(S,A,L,O):=lll(M:=[[2,1],[1,2]])
```

Output :

```
[[-1,1],[2,1]], [[-1,1],[1,0]], [[1,0],[1/-2,1]],
                [[-1,1],[3/2,3/2]]
```

Hence :
S=[[-1,1],[2,1]]
A=[[-1,1],[1,0]]
L=[[1,0],[1/-2,1]]
O=[[-1,1],[3/2,3/2]]
Hence the original basis is v1=[2,1], v2=[1,2]
and the short basis is w1=[-1,1], w2=[2,1].
Since w1=-v1+v2 and w2=v1 then :
A:=[[-1,1],[1,0]], A*M==S and L*O==S.
Input :

```
(S,A,L,O):=lll([[3,2,1],[1,2,3],[2,3,1]])
```

Output :

```
S=[[-1,1,0],[-1,-1,2],[3,2,1]]

A= [[-1,0,1],[0,1,-1],[1,0,0]]

L= [[1,0,0],[0,1,0],[(-1)/2,(-1)/2,1]]

O= [[-1,1,0],[-1,-1,2],[2,2,2]]
```

Input :
M:=[[3,2,1],[1,2,3],[2,3,1]]
Properties :
A*M==S and L*O==S

5.51 Quadratic forms

5.51.1 Matrix of a quadratic form : q2a

q2a takes two arguments : the symbolic expression of a quadratic form q and a vector of variable names.
q2a returns the matrix A of q.
Input :

```
q2a(2*x*y,[x,y])
```

Output :

```
[[0,1],[1,0]]
```

5.51.2 Transform a matrix into a quadratic form : a2q

a2q takes two arguments : the symmetric matrix A of a quadratic form q and a vector of variable names of the same size.
a2q returns the symbolic expression of the quadratic form q.
Input :

```
a2q([[0,1],[1,0]],[x,y])
```

Output :

```
2*x*y
```

Input :

```
a2q([[1,2],[2,4]],[x,y])
```

Output :

```
x^2+4*x*y+4*y^2
```

5.51.3 Reduction of a quadratic form : gauss

gauss takes two arguments : a symbolic expression representing a quadratic form q and a vector of variable names.
gauss returns q written as sum or difference of squares using Gauss algorithm.
Input :

```
gauss(2*x*y,[x,y])
```

Output :

```
(y+x)^2/2+(-(y-x)^2)/2
```

5.51.4 Gram-Schmidt orthonormalization : gramschmidt

gramschmidt takes one or two arguments :

- a matrix viewed as a list of row vectors, the scalar product being the canonical scalar product, or
- a list of elements that is a basis of a vector subspace, and a function that defines a scalar product on this vector space.

gramschmidt returns an orthonormal basis for this scalar product.
Input :

```
normal(gramschmidt([[1,1,1],[0,0,1],[0,1,0]]))
```

Or input :

```
normal(gramschmidt([[1,1,1],[0,0,1],[0,1,0]],dot))
```

Output :

```
[[(sqrt(3))/3,(sqrt(3))/3,(sqrt(3))/3],[(-(sqrt(6)))/6,
(-(sqrt(6)))/6,(sqrt(6))/3],[(-(sqrt(2)))/2,(sqrt(2))/2,0]]
```

Example
We define a scalar product on the vector space of polynomials by:

$$P \cdot Q = \int_{-1}^{1} P(x)Q(x)dx$$

Input :

```
gramschmidt([1,1+x],(p,q)->integrate(p*q,x,-1,1))
```

Or define the function p_scal, input :
`p_scal(p,q):=integrate(p*q,x,-1,1)`
then input :

```
gramschmidt([1,1+x],p_scal)
```

Output :

```
[1/(sqrt(2)),(1+x-1)/sqrt(2/3)]
```

5.51.5 Graph of a conic : conique

conique takes as argument the equation of a conic with respect to x, y. You may also specify the names of the variables as second and third arguments or as a vector as second argument.
conique draws this conic.
Input :

```
conique(2*x^2+2*x*y+2*y^2+6*x)
```

Output :

```
the graph of the ellipsis of center -2+i and equation
2*x^2+2*x*y+2*y^2+6*x=0
```

Remark :
See also conique_reduite for the parametric equation of the conic.

5.51.6 Conic reduction : `conique_reduite`

`conique_reduite` takes two arguments : the equation of a conic and a vector of variable names.
`conique_reduite` returns a list whose elements are:

- the origin of the conic,
- the matrix of a basis in which the conic is reduced,
- 0 or 1 (0 if the conic is degenerate),
- the reduced equation of the conic
- a vector of its parametric equations.

Input :

```
conique_reduite(2*x^2+2*x*y+2*y^2+5*x+3,[x,y])
```

Output :

```
[[-5/3,5/6],[[-1/(sqrt(2)),1/(sqrt(2))],[-1/(sqrt(2)),
    -1/(sqrt(2))]],1,3*x^2+y^2+-7/6,[[(-10+5*i)/6+
  (1/(sqrt(2))+(i)/(sqrt(2)))*((sqrt(14)*cos(` t`))/6+
  ((i)*sqrt(42)*sin(` t`))/6),` t`,0,2*pi,(2*pi)/60]]]
```

Which means that the conic is not degenerate, its reduced equation is

$$3x^2 + y^2 - 7/6 = 0$$

its origin is $-5/3+5*i/6$, its axes are parallel to the vectors $(-1, 1)$ and $(-1, -1)$. Its parametric equation is

$$\frac{-10 + 5 * i}{6} + \frac{(1 + i)}{\sqrt{2}} * \frac{(\sqrt{14} * cos(t) + i * \sqrt{42} * sin(t))}{6}$$

where the suggested parameter values for drawing are t from 0 to 2π with `tstep`= $2\pi/60$.

Remark :
Note that if the conic is degenerate and is made of 1 or 2 line(s), the lines are not given by their parametric equation but by the list of two points of the line.
Input :

```
conique_reduite(x^2-y^2+3*x+y+2)
```

Output :

```
[[(-3)/2,1/2],[[1,0],[0,1]],0,x^;2-y^2,
    [[(-1+2*i)/(1-i),(1+2*i)/(1-i)],
     [(-1+2*i)/(1-i),(-1)/(1-i)]]]
```

5.51.7 Graph of a quadric : quadrique

quadrique takes as arguments the expression of a quadric with respect to x, y, z. You may also specify the variables as a vector (second argument) or as second, third and fourth arguments.
quadrique draws this quadric.
Input :

```
quadrique(7*x^2+4*y^2+4*z^2+4*x*y-
    4*x*z-2*y*z-4*x+5*y+4*z-18)
```

Output :

```
the drawing of the ellipsoid of equation
7*x^2+4*y^2+4*z^2+4*x*y-4*x*z-2*y*z-4*x+5*y+4*z-18=0
```

See also quadrique_reduite for the parametric equation of the quadric.

5.51.8 Quadric reduction : quadrique_reduite

quadrique_reduite takes two arguments : the equation of a quadric and a vector of variable names.
quadrique_reduite returns a list whose elements are:

- the origin,
- the matrix of a basis where the quadric is reduced,
- 0 or 1 (0 if the quadric is degenerate),
- the reduced equation of the quadric
- a vector with its parametric equations.

Warning ! u, v will be used as parameters of the parametric equations : these variables should not be assigned (purge them before calling quadrique_reduite).
Input :

```
quadrique_reduite(7*x^2+4*y^2+4*z^2+
  4*x*y-4*x*z-2*y*z-4*x+5*y+4*z-18)
```

Output is a list containing :

- The origin (center of symmetry) of the quadric

```
[11/27,(-26)/27,(-29)/54],
```

- The matrix of the basis change:

```
[[(sqrt(6))/3,(sqrt(5))/5,(-(sqrt(30)))/15],
        [(sqrt(6))/6,0,(sqrt(30))/6],
[(-(sqrt(6)))/6,(2*sqrt(5))/5,(sqrt(30))/30]],
```

- 1 hence the quadric is not degenerated

- the reduced equation of the quadric :

```
0,9*x^2+3*y^2+3*z^2+(-602)/27,
```

- The parametric equations (in the original frame) are :

```
[[(sqrt(6)*sqrt(602/243)*sin(u)*cos(v))/3+
(sqrt(5)*sqrt(602/81)*sin(u)*sin(v))/5+
((-(sqrt(30)))*sqrt(602/81)*cos(u))/15+11/27,
(sqrt(6)*sqrt(602/243)*sin(u)*cos(v))/6+
(sqrt(30)*sqrt(602/81)*cos(u))/6+(-26)/27,
((-(sqrt(6)))*sqrt(602/243)*sin(u)*cos(v))/6+
(2*sqrt(5)*sqrt(602/81)*sin(u)*sin(v))/5+
(sqrt(30)*sqrt(602/81)*cos(u))/30+(-29)/54], u=(0
.. pi),v=(0.. (2*pi)),ustep=(pi/20),
vstep=((2*pi)/20)]]
```

Hence the quadric is an ellipsoid and its reduced equation is :

$$9*x^2+3*y^2+3*z^2+(-602)/27=0$$

after the change of origin $[11/27,(-26)/27,(-29)/54]$, the matrix of basis change `P` is :

$$\begin{bmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{5}}{5} & -\frac{\sqrt{30}}{15} \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} \\ -\frac{\sqrt{6}}{6} & \frac{2\sqrt{5}}{5} & \frac{\sqrt{30}}{30} \end{bmatrix}$$

Its parametric equation is :

$$\left\{\begin{array}{l} x=\frac{\sqrt{6}\sqrt{\frac{602}{243}}\sin(u)\cos(v)}{3}+\frac{\sqrt{5}\sqrt{\frac{602}{81}}\sin(u)\sin(v)}{5}-\frac{\sqrt{30}\sqrt{\frac{602}{81}}\cos(u)}{15}+\frac{11}{27} \\ y=\frac{\sqrt{6}\sqrt{\frac{602}{243}}\sin(u)\cos(v)}{6}+\frac{\sqrt{30}\sqrt{\frac{602}{81}}\cos(u))}{6}-\frac{26}{27} \\ z=\frac{-\sqrt{6}\sqrt{\frac{602}{243}}*\sin(u)\cos(v)}{6}+\frac{2\sqrt{5}\sqrt{\frac{602}{81}}\sin(u)\sin(v)}{5}+\frac{\sqrt{30}\sqrt{\frac{602}{81}}\cos(u)}{30}-\frac{29}{54} \end{array}\right.$$

Remark :
Note that if the quadric is degenerate and made of 1 or 2 plane(s), each plane is not given by its parametric equation but by the list of a point of the plane and of a normal vector to the plane.
Input :

```
quadrique_reduite(x^2-y^2+3*x+y+2)
```

Output :

```
[[(-3)/2,1/2,0],[[1,0,0],[0,1,0],[0,0,-1]],0,x^2-y^2,
[hyperplan([1,1,0],[(-3)/2,1/2,0]),
hyperplan([1,-1,0],[(-3)/2,1/2,0])]]
```

5.52 Multivariate calculus

5.52.1 Gradient : `derive deriver diff grad`

`derive` (or `diff` or `grad`) takes two arguments : an expression F of n real variables and a vector of these variable names.
`derive` returns the gradient of F, where the gradient is the vector of all partial derivatives, for example in dimension $n = 3$

$$\overrightarrow{\text{grad}}(F) = [\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}]$$

Example
Find the gradient of $F(x, y, z) = 2x^2y - xz^3$.
Input :

```
derive(2*x^2*y-x*z^3,[x,y,z])
```

or :

```
diff(2*x^2*y-x*z^3,[x,y,z])
```

or :

```
grad(2*x^2*y-x*z^3,[x,y,z])
```

Output :

```
[2*2*x*y-z^3,2*x^2,-(x*3*z^2)]
```

Output after simplification with `normal(ans())` :

```
[4*x*y-z^3,2*x^2,-(3*x*z^2)]
```

To find the critical points of $F(x, y, z) = 2x^2y - xz^3$, input :

```
solve(derive(2*x^2*y-x*z^3,[x,y,z]),[x,y,z])
```

Output :

```
[[0,y,0]]
```

5.52.2 Laplacian : `laplacian`

`laplacian` takes two arguments : an expression F of n real variables and a vector of these variable names.
`laplacian` returns the Laplacian of$F$, that is the sum of all second partial derivatives, for example in dimension $n = 3$:

$$\nabla^2(F) = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

Example
Find the Laplacian of $F(x, y, z) = 2x^2y - xz^3$.
Input :

```
laplacian(2*x^2*y-x*z^3,[x,y,z])
```

Output :

```
4*y+-6*x*z
```

5.52.3 Hessian matrix : hessian

`hessian` takes two arguments : an expression F of n real variables and a vector of these variable names.
`hessian` returns the hessian matrix of F, that is the matrix of the derivatives of order 2.
Example
Find the hessian matrix of $F(x, y, z) = 2x^2y - xz^3$.
Input :

```
hessian(2*x^2*y-x*z^3 , [x,y,z])
```

Output :

```
[[4*y,4*x,-(3*z^2)],[2*2*x,0,0],[-(3*z^2),0,x*3*2*z]]
```

To have the hessian matrix at the critical points, first input :

```
solve(derive(2*x^2*y-x*z^3,[x,y,z]),[x,y,z])
```

Output is the critical points :

```
[[0,y,0]]
```

Then, to have the hessian matrix at this points, input :

```
subst([[4*y,4*x,-(3*z^2)],[2*2*x,0,0],
[-(3*z^2),0,6*x*z]],[x,y,z],[0,y,0])
```

Output :

```
[[4*y,4*0,-(3*0^2)],[4*0,0,0],[-(3*0^2),0,6*0*0]]
```

and after simplification :

```
[[4*y,0,0],[0,0,0],[0,0,0]]
```

5.52.4 Divergence : divergence

`divergence` takes two arguments : a vector field of dimension n depending on n real variables.
`divergence` returns the divergence of F that is the sum of the derivative of the k-th component with respect to the k-th variable. For example in dimension $n = 3$:

$$\texttt{divergence([A,B,C],[x,y,z])}=\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$$

Input :

```
divergence([x*z,-y^2,2*x^y],[x,y,z])
```

Output :

```
z+-2*y
```

5.52.5 Rotational : `curl`

`curl` takes two arguments : a 3-d vector field depending on 3 variables.
`curl` returns the rotational of the vector, defined by:

`curl([A,B,C],[x,y,z])=`$[\frac{\partial C}{\partial y}-\frac{\partial B}{\partial z},\frac{\partial A}{\partial z}-\frac{\partial C}{\partial x},\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}]$

Note that n **must be equal to 3**.
Input :

```
curl([x*z,-y^2,2*x^y],[x,y,z])
```

Output :

```
[2*x^y*log(x),x-2*y*x^(y-1),0]
```

5.52.6 Potential : `potential`

`potential` takes two arguments : a vector field $\overrightarrow{V}$ in R^n with respect to n real variables and the vector of these variable names.
`potential` returns, if it is possible, a function U such that $\overrightarrow{\text{grad}}(U)=\overrightarrow{V}$. When it is possible, we say that $\overrightarrow{V}$ derives the potential U, and U is defined up to a constant.
`potential` is the reciprocal function of `derive`.
Input :

```
potential([2*x*y+3,x^2-4*z,-4*y],[x,y,z])
```

Output :

```
2*y*x^2/ 2+3*x+(x^2-4*z-2*x^2/2)*y
```

Note that in $\mathbb{R}^3$ a vector $\overrightarrow{V}$ is a gradient if and only if its rotational is zero i.e. if `curl(V)=0`. In time-independent electro-magnetism, $\overrightarrow{V}$=$\overrightarrow{E}$ is the electric field and U is the electric potential.

5.52.7 Conservative flux field : `vpotential`

`vpotential` takes two arguments : a vector field $\overrightarrow{V}$ in R^n with respect to n real variables and the vector of these variable names.
`vpotential` returns, if it is possible, a vector $\overrightarrow{U}$ such that $\overrightarrow{\text{curl}}(\overrightarrow{U})=\overrightarrow{V}$. When it is possible we say that $\overrightarrow{V}$ is a conservative flux field or a solenoidal field. The general solution is the sum of a particular solution and of the gradient of an arbitrary function, `Xcas` returns a particular solution with zero as first component.
`vpotential` is the reciprocal function of `curl`.
Input :

```
vpotential([2*x*y+3,x^2-4*z,-2*y*z],[x,y,z])
```

Output :

```
[0,(-(2*y))*z*x,-x^3/3-(-(4*z))*x+3*y]
```

In $\mathbb{R}^3$, a vector field $\overrightarrow{V}$ is a rotational if and only if its divergence is zero (`divergence(V,[x,y,z])=0`). In time-independent electro-magnetism, $\overrightarrow{V}$= $\overrightarrow{B}$ is the magnetic field and $\overrightarrow{U}$= $\overrightarrow{A}$ is the potential vector.

5.53 Equations

5.53.1 Define an equation : `equal`

`equal` takes as argument the two members of an equation.
`equal` returns this equation. It is the prefixed version of =
Input :

```
equal(2x-1,3)
```

Output :

```
(2*x-1)=3
```

We can also directly write `(2*x-1)=3`.

5.53.2 Transform an equation into a difference : `equal2diff`

`equal2diff` takes as argument an equation.
`equal2diff` returns the difference of the two members of this equation.
Input :

```
equal2diff(2x-1=3)
```

Output :

```
2*x-1-3
```

5.53.3 Transform an equation into a list : `equal2list`

`equal2list` takes as argument an equation.
`equal2list` returns the list of the two members of this equation.
Input :

```
equal2list(2x-1=3)
```

Output :

```
[2*x-1,3]
```

5.53.4 The left member of an equation : `left gauche lhs`

`left` or `lhs` takes as argument an equation or an interval.
`left` or `lhs` returns the left member of this equation or the left bound of this interval.
Input :

```
left(2x-1=3)
```

Or input:

```
lhs(2x-1=3)
```

Output :

```
2*x-1
```

Input :

```
left(1..3)
```

Or input:

```
lhs(1..3)
```

Output :

```
1
```

5.53.5 The right member of an equation : `right droit rhs`

`right` or `rhs` takes as argument an equation or an interval.
`right` or `rhs` returns the right member of this equation or the right bound of this interval.
Input :

```
right(2x-1=3)
```

or :

```
rhs(2x-1=3)
```

Output :

```
3
```

Input :

```
right(1..3)
```

or :

```
rhs(1..3)
```

Output :

```
3
```

5.53.6 Solving equation(s): `solve`

`solve` solves an equation or a system of polynomial equations. It takes 2 arguments:

- Solving an equation
  `solve` takes as arguments an equation between two expressions or an expression (`=0` is omitted), and a variable name (by default `x`).
  `solve` solves this equation.

- Solving a system of polynomial equations
  `solve` takes as arguments two vectors : a vector of polynomial equations and a vector of variable names.
  `solve` solves this polynomial equation system.

Remarks:

- In real mode, `solve` returns only real solutions. To have the complex solutions, switch to complex mode, e.g. by checking `Complex` in the cas configuration, or use the `cSolve` command.
- For trigonometric equations, `solve` returns by default the principal solutions. To have all the solutions check `All_trig_sol` in the cas configuration.

Examples :

- Solve $x^4 - 1 = 3$
  Input :

  ```
  solve(x^4-1=3)
  ```

  Output in real mode :

  ```
  [sqrt(2),-(sqrt(2))]
  ```

  Output in complex mode :

  ```
  [sqrt(2),-(sqrt(2)),(i)*sqrt(2),-((i)*sqrt(2))]
  ```

- Solve $\exp(x) = 2$
  Input :

  ```
  solve(exp(x)=2)
  ```

  Output in real mode :

  ```
  [log(2)]
  ```

- Find x, y such that $x + y = 1, x - y = 0$
  Input :

  ```
  solve([x+y=1,x-y],[x,y])
  ```

  Output :

  ```
  [[1/2,1/2]]
  ```

- Find x, y such that $x^2 + y = 2, x + y^2 = 2$
  Input :

  ```
  solve([x^2+y=2,x+y^2=2],[x,y])
  ```

  Output :

  ```
  [[-2,-2],[1,1],[(-sqrt(5)+1)/2,(1+sqrt(5))/2],
  ```

```
[(sqrt(5)+1)/2,(1-sqrt(5))/2]]
```

- Find x, y, z such that $x^2 - y^2 = 0, x^2 - z^2 = 0$
  Input :

  ```
  solve([x^2-y^2=0,x^2-z^2=0],[x,y,z])
  ```

  Output :

  ```
  [[x,x,x],[x,-x,-x],[x,-x,x],[x,x,-x]]
  ```

- Solve $\cos(2 * x) = 1/2$
  Input :

  ```
  solve(cos(2*x)=1/2)
  ```

  Output :

  ```
  [pi/6,(-pi)/6]
  ```

  Output with `All_trig_sol` checked :

  ```
  [(6*pi*n_0+pi)/6,(6*pi*n_0-pi)/6]
  ```

- Find the intersection of a straight line (given by a list of equations) and a plane.
  For example, let D be the straight line of cartesian equations $[y - z = 0, z - x = 0]$ and let P the plane of equation $x-1+y+z = 0$. Find the intersection of D and P.
  Input :

  ```
  solve([[y-z=0,z-x=0],x-1+y+z=0],[x,y,z])
  ```

  Output :

  ```
  [[1/3,1/3,1/3]]
  ```

5.53.7 Equation solving in $\mathbb{C}$: `cSolve`

`cSolve` takes two arguments and solves an equation or a system of polynomial equations.

- solving an equation
  `cSolve` takes as arguments an equation between two expressions or an expression (`=0` is omitted), and a variable name (by default `x`).
  `cSolve` solves this equation in $\mathbb{C}$ even if you are in real mode.

- solving a system of polynomial equations
  `cSolve` takes as arguments two vectors : a vector of polynomial equations and a vector of variable names.
  `cSolve` solves this equation system in $\mathbb{C}$ even if you are in real mode.

Input :

```
cSolve(x^4-1=3)
```

Output :

```
[sqrt(2),-(sqrt(2)),(i)*sqrt(2),-((i)*sqrt(2))]
```

Input :

```
cSolve([-x^2+y=2,x^2+y],[x,y])
```

Output :

```
[[i,1],[-i,1]]
```

5.54 Linear systems

In this paragraph, we call the "augmented matrix" of the system $A \cdot X = B$ (or matrix "representing" the system $A{\cdot}X = B$), the matrix obtained by gluing the column vector B or $-B$ to the right of the matrix A, as with `border(A,tran(B))`.

5.54.1 Matrix of a system : `syst2mat`

`syst2mat` takes two vectors as arguments. The components of the first vector are the equations of a linear system and the components of the second vector are the variable names.
`syst2mat` returns the augmented matrix of the system $AX = B$, obtained by gluing the column vector $-B$ to the right of the matrix A.
Input :

```
syst2mat([x+y,x-y-2],[x,y])
```

Output :

```
[[1,1,0],[1,-1,-2]]
```

Input :

```
syst2mat([x+y=0,x-y=2],[x,y])
```

Output :

```
[[1,1,0],[1,-1,-2]]
```

Warning !!!
The variables (here `x` and `y`) must be purged.

5.54.2 Gauss reduction of a matrix : `ref`

`ref` is used to solve a linear system of equations written in matrix form:

```
A*X=B
```

The argument of `ref` is the augmented matrix of the system (the matrix obtained by augmenting the matrix `A` to the right with the column vector `B`).
The result is a matrix `[A1,B1]` where `A1` has zeros under its principal diagonal, and the solutions of:

```
A1*X=B1
```

are the same as the solutions of:

```
A*X=B
```

For example, solve the system :

$$\begin{cases} 3x+y &= -2 \\ 3x+2y &= 2 \end{cases}$$

Input :

```
ref([[3,1,-2],[3,2,2]])
```

Output :

```
[[1,1/3,-2/3],[0,1,4]]
```

Hence the solution is $y = 4$ (last row) and $x = -2$ (substitute y in the first row).

5.54.3 Gauss-Jordan reduction: `rref gaussjord`

`rref` solves a linear system of equations written in matrix form (see also 5.32.17) :

```
A*X=B
```

`rref` takes one or two arguments.

- If `rref` has only one argument, this argument is the augmented matrix of the system (the matrix obtained by augmenting matrix `A` to the right with the column vector `B`).
  The result is a matrix `[A1,B1]` : `A1` has zeros both above and under its principal diagonal and has 1 on its principal diagonal, and the solutions of:

  ```
  A1*X=B1
  ```

  are the same as :

  ```
  A*X=B
  ```

For example, to solve the system:

$$\left\{ \begin{array}{rcr} 3x+y & = & -2 \\ 3x+2y & = & 2 \end{array} \right.$$

Input :

```
rref([[3,1,-2],[3,2,2]])
```

Output :

```
[[1,0,-2],[0,1,4]]
```

Hence $x=-2$ and $y=4$ is the solution of this system.

`rref` can also solve several linear systems of equations having the same first member. We write the second members as a column matrix.
Input :

```
rref([[3,1,-2,1],[3,2,2,2]])
```

Output :

```
[[1,0,-2,0],[0,1,4,1]]
```

Which means that ($x=-2$ and $y=4$) is the solution of the system

$$\left\{ \begin{array}{rcr} 3x+y & = & -2 \\ 3x+2y & = & 2 \end{array} \right.$$

and ($x=0$ and $y=1$) is the solution of the system

$$\left\{ \begin{array}{rcr} 3x+y & = & 1 \\ 3x+2y & = & 2 \end{array} \right.$$

- If `rref` has two parameters, the second parameter must be an integer k, and the Gauss-Jordan reduction will be performed on (at most) the first k columns.
  Input :

  ```
  rref([[3,1,-2,1],[3,2,2,2]],1)
  ```

  Output :

  ```
  [[3,1,-2,1],[0,1,4,1]]
  ```

5.54.4 Solving A*X=B : `simult`

`simult` is used to solve a linear system of equations (resp. several linear systems of equations with the same matrix `A`) written in matrix form (see also 5.32.17) :

```
A*X=b (resp.  A*X=B)
```

`simult` takes as arguments the matrix `A` of the system and the column vector (i.e. a one column matrix) `b` of the second member of the system (resp. the matrix `B` whose columns are the vectors `b` of the second members of the different systems). The result is a column vector solution of the system (resp. a matrix whose columns are the solutions of the different systems).
For example, to solve the system :

$$\begin{cases} 3x + y & = & -2 \\ 3x + 2y & = & 2 \end{cases}$$

Input :

```
simult([[3,1],[3,2]],[[-2],[2]])
```

Output :

```
[[-2],[4]]
```

Hence $x = -2$ and $y = 4$ is the solution.
Input :

```
simult([[3,1],[3,2]],[[-2,1],[2,2]])
```

Output :

```
[[-2,0],[4,1]]
```

Hence $x = -2$ and $y = 4$ is the solution of

$$\begin{cases} 3x + y & = & -2 \\ 3x + 2y & = & 2 \end{cases}$$

whereas $x = 0$ and $y = 1$ is the solution of

$$\begin{cases} 3x + y & = & 1 \\ 3x + 2y & = & 2 \end{cases}$$

5.54.5 Step by step Gauss-Jordan reduction of a matrix : `pivot`

`pivot` takes three arguments : a matrix with n rows and p columns and two integers l and c such that $0 \leq l < n$, $0 \leq c < p$ and $A_{l,c} \neq 0$.
`pivot(A,l,c)` performs one step of the Gauss-Jordan method using `A[l,c]` as pivot and returns an equivalent matrix with zeros in the column `c` of `A` (except at row l).
Input :

```
pivot([[1,2],[3,4],[5,6]],1,1)
```

Output :

```
[[-2,0],[3,4],[2,0]]
```

Input :

```
pivot([[1,2],[3,4],[5,6]],0,1)
```

Output :

```
[[1,2],[2,0],[4,0]]
```

5.54.6 Linear system solving: `linsolve`

`linsolve` is used to solve a system of linear equations.
`linsolve` has two arguments: a list of equations or expressions (in that case the convention is that the equation is $expression = 0$), and a list of variable names.
`linsolve` returns the solution of the system in a list.
Input :

```
linsolve([2*x+y+z=1,x+y+2*z=1,x+2*y+z=4],[x,y,z])
```

Output :

```
[1/-2,5/2,1/-2]
```

Which means that

$$x = -\frac{1}{2}, y = \frac{5}{2}, z = -\frac{1}{2}$$

is the solution of the system :

$$\begin{cases} 2x + y + z & = 1 \\ x + y + 2z & = 1 \\ x + 2y + z & = 4 \end{cases}$$

5.54.7 Finding linear recurrences : `reverse_rsolve`

`reverse_rsolve` takes as argument a vector $v = [v_0 ... v_{2n-1}]$ made of the first $2n$ terms of a sequence (v_n) which is supposed to verify a linear recurrence relation of degree smaller than n

$$x_n * v_{n+k} + ... + x_0 * v_k = 0$$

where the x_j are $n + 1$ unknowns.
`reverse_rsolve` returns the list $x = [x_n, ..., x_0]$ of the x_j coefficients (if $x_n \neq 0$ it is reduced to 1).

In other words `reverse_rsolve` solves the linear system of n equations :

$$\begin{aligned} x_n * v_n + ... + x_0 * v_0 &= 0 \\ &... \\ x_n * v_{n+k} + ... + x_0 * v_k &= 0 \\ &... \\ x_n * v_{2*n-1} + ... + x_0 * v_{n-1} &= 0 \end{aligned}$$

The matrix A of the system has n rows and $n+1$ columns :

$$A = [[v_0, v_1 ... v_n], [v_1, v_2, ... v_{n-1}], ..., [v_{n-1}, v_n ... v_{2n-1}]]$$

`reverse_rsolve` returns the list $x = [x_n, ... x_1, x_0]$ with $x_n = 1$ and x is the solution of the system $A * \texttt{revlist}(x)$.

Examples

- Find a sequence satisfying a linear recurrence of degree at most 2 whose first elements 1, -1, 3, 3.
  Input :

  ```
  reverse_rsolve([1,-1,3,3])
  ```

  Output :

  ```
  [1,-3,-6]
  ```

  Hence $x_0 = -6$, $x_1 = -3$, $x_2 = 1$ and the recurrence relation is

  $$v_{k+2} - 3v_{k+1} - 6v_k = 0$$

  Without `reverse_rsolve`, we would write the matrix of the system :
  `[[1,-1,3],[-1,3,3]]` and use the `rref` command :
  `rref([[1,-1,3],[-1,3,3]])`
  Output is `[[1,0,6],[0,1,3]]` hence $x_0 = -6$ and $x_1 = -3$ (because $x_2 = 1$).

- Find a sequence satisfying a linear recurrence of degree at most 3 whose first elements are 1, -1, 3, 3,-1, 1.
  Input :

  ```
  reverse_rsolve([1,-1,3,3,-1,1])
  ```

  Output :

  ```
  [1,(-1)/2,1/2,-1]
  ```

  Hence so, $x_0 = -1$, $x_1 = 1/2$, $x_2 = -1/2$, $x_3 = 1$, the recurrence relation is

  $$v_{k+3} - \frac{1}{2}v_{k+2} + \frac{1}{2}v_{k+1} - v_k = 0$$

  Without `reverse_rsolve`, we would write the matrix of the system :
  `[[1,-1,3,3],[-1,3,3,-1],[3,3,-1,1]]`.
  Using `rref` command, we would input :
  `rref([[1,-1,3,3],[-1,3,3,-1],[3,3,-1,1]])`
  Output is `[1,0,0,1],[0,1,0,1/-2],[0,0,1,1/2]]` hence $x_0 = -1$, $x_1 = 1/2$ and $x_2 = -1/2$ because $x_3 = 1$),

5.55 Differential equations

This section is limited to symbolic (or exact) solutions of differential equations. For numeric solutions of differential equations, see `odesolve`. For graphic representation of solutions of differential equations, see `plotfield`, `plotode` and `interactive_plotode`.

5.55.1 Solving differential equations : `desolve deSolve dsolve`

`desolve` (or `deSolve`) can solve :

- linear differential equations with constant coefficients,
- first order linear differential equations,
- first order differential equations without y,
- first order differential equations without x,
- first order differential equations with separable variables,
- first order homogeneous differential equations ($y' = F(y/x)$),
- first order differential equations with integrating factor,
- first order Bernoulli differential equations ($a(x)y' + b(x)y = c(x)y^n$),
- first order Clairaut differential equations ($y = x * y' + f(y')$).

`desolve` takes as arguments :

- if the independent variable is the current variable (here supposed to be x),
  - the differential equation (or the list of the differential equation and of the initial conditions)
  - the unknown (usually `y`).

  In the differential equation, the function y is denoted by y, its first derivative $y\prime$ is denoted by y′, and its second derivative y'' is written y″.
  For example `desolve(y″+2*y′+y,y)` or
  `desolve([y″+2*y′+y,y(0)=1,y′(0)=0],y)`.

- if the independent variable is not the current variable, for example t instead of x,
  - the differential equation (or the list of the differential equation and of the initial conditions),
  - the variable, e.g. `t`
  - the unknown as a variable `y` or as a function `y(t)`.

In the differential equation, the function y is denoted by $y(t)$, its derivative $y\prime$ is denoted by `diff(y(t),t)`, and its second derivative y'' is denoted by `diff(y(t),t$2)`.
For example :
`desolve(diff(y(t),t$2)+2*diff(y(t),t)+y(t),y(t));` or
`desolve(diff(y(t),t$2)+2*diff(y(t),t)+y(t),t,y);` and

```
desolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t),
         y(0)=1,y'(0)=0],y(t)); or
desolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t),
         y(0)=1,y'(0)=0],t,y);
```

If there is no initial conditions (or one initial condition for a second order equation), `desolve` returns the general solution in terms of constants of integration `c_0`, `c_1`, where `y(0)=c_0` and `y'(0)=c_1`, or a list of solutions.
Examples

- Examples of second linear differential equations with constant coefficients.

  1. Solve :
  $$y'' + y = \cos(x)$$
  Input (typing twice prime for `y''`):
  ```
  desolve(y''+y=cos(x),y)
  ```
  or input :
  ```
  desolve((diff(diff(y))+y)=(cos(x)),y)
  ```
  Output :
  ```
  c_0*cos(x)+(x+2*c_1)*sin(x)/2
  ```
  `c_0, c_1` are the constants of integration : `y(0)=c_0` and `y'(0)=c_1`.
  If the variable is not `x` but `t`, input :
  ```
  desolve(derive(derive(y(t),t),t)+y(t)=cos(t),t,y)
  ```
  Output :
  ```
  c_0*cos(t)+(t+2*c_1)/2*sin(t)
  ```
  `c_0, c_1` are the constants of integration : `y(0)=c_0` and `y'(0)=c_1`.

  2. Solve :
  $$y'' + y = \cos(x), \quad y(0) = 1$$
  Input :
  ```
  desolve([y''+y=cos(x),y(0)=1],y)
  ```
  Output :
  ```
  [cos(x)+(x+2*c_1)/2*sin(x)]
  ```

the components of this vector are solutions (here there is just one component, so we have just one solution depending of the constant c_1).

3. Solve :

$$y'' + y = \cos(x) \quad (y(0))^2 = 1$$

Input :

```
desolve([y''+y=cos(x),y(0)^2=1],y)
```

Output :

```
[-cos(x)+(x+2*c_1)/2*sin(x),cos(x)+(x+2*c_1)/2*sin(x)]
```

each component of this list is a solution, we have two solutions depending on the constant c_1 ($y'(0) = c_1$) and corresponding to $y(0) = 1$ and to $y(0) = -1$.

4. Solve :

$$y'' + y = \cos(x), \quad (y(0))^2 = 1 \quad y'(0) = 1$$

Input :

```
desolve([y''+y=cos(x),y(0)^2=1,y'(0)=1],y)
```

Output :

```
[-cos(x)+(x+2)/2*sin(x),cos(x)+(x+2)/2*sin(x)]
```

each component of this list is a solution (we have two solutions).

5. Solve :

$$y'' + 2y' + y = 0$$

Input :

```
desolve(y''+2*y'+y=0,y)
```

Output :

```
(x*c_0+x*c_1+c_0)*exp(-x)
```

the solution depends of 2 constants of integration : c_0, c_1 (y(0)=c_0 and y'(0)=c_1).

6. Solve :

$$y'' - 6y' + 9y = xe^{3x}$$

Input:

```
desolve(y''-6*y'+9*y=(x*exp(3*x),y)
```

Output :

```
(x^3+(-(18*x))*c_0+6*x*c_1+6*c_0)*1/6*exp(3*x)
```

the solution depends on 2 constants of integration : c_0, c_1 (y(0)=c_0 and y'(0)=c_1).

- Examples of first order linear differential equations.

1. Solve :
$$xy' + y - 3x^2 = 0$$
Input :
```
desolve(x*y'+y-3*x^2,y)
```
Output :
```
(3*1/3*x^3+c_0)/x
```
2. Solve :
$$y' + x * y = 0, y(0) = 1$$
Input :
```
desolve([y'+x*y=0, y(0)=1]),y)
```
or :
```
desolve((y'+x*y=0) && (y(0)=1),y)
```
Output :
```
[1/(exp(1/2*x^2))]
```
3. Solve :
$$x(x^2 - 1)y' + 2y = 0$$
Input :
```
desolve(x*(x^2-1)*y'+2*y=0,y)
```
Output :
```
(c_0)/((x^2-1)/(x^2))
```
4. Solve :
$$x(x^2 - 1)y' + 2y = x^2$$
Input :
```
desolve(x*(x^2-1)*y'+2*y=x^2,y)
```
Output :
```
(ln(x)+c_0)/((x^2-1)/(x^2))
```
5. If the variable is t instead of x, for example :
$$t(t^2 - 1)y'(t) + 2y(t) = t^2$$
Input :
```
desolve(t*(t^2-1)*diff(y(t),t)+2*y(t)=(t^2),y(t))
```
Output :
```
(ln(t)+c_0)/((t^2-1)/(t^2))
```

6. Solve :

$$x(x^2-1)y'+2y=x^2, y(2)=0$$

Input :

```
desolve([x*(x^2-1)*y'+2*y=x^2,y(0)=1],y)
```

Output :

```
[(ln(x)-ln(2))*1/(x^2-1)*x^2]
```

7. Solve :

$$\sqrt{1+x^2}y'-x-y=\sqrt{1+x^2}$$

Input :

```
desolve(y'*sqrt(1+x^2)-x-y-sqrt(1+x^2),y)
```

Output :

```
(-c_0+ln(sqrt(x^2+1)-x))/(x-sqrt(x^2+1))
```

- Examples of first differential equations with separable variables.

1. Solve :

$$y'=2\sqrt{y}$$

Input :

```
desolve(y'=2*sqrt(y),y)
```

Output :

```
[x^2+-2*x*c_0+c_0^2]
```

2. Solve :

$$xy'\ln(x)-y(3\ln(x)+1)=0$$

Input :

```
desolve(x*y'*ln(x)-(3*ln(x)+1)*y,y)
```

Output :

```
c_0*x^3*ln(x)
```

- Examples of Bernoulli differential equations $a(x)y'+b(x)y=c(x)y^n$ where n is a real constant.
  The method used is to divide the equation by y^n, so that it becomes a first order linear differential equation in $u=1/y^{n-1}$.

1. Solve :

$$xy'+2y+xy^2=0$$

Input :

```
desolve(x*y'+2*y+x*y^2,y)
```

Output :

```
[1/(exp(2*ln(x))*(-1/x+c_0))]
```

2. Solve :

$$xy' - 2y = xy^3$$

Input :

```
desolve(x*y'-2*y-x*y^3,y)
```

Output :

```
[((-2*1/5*x^5+c_0)*exp(-(4*log(x))))^(1/-2),

-((-2*1/5*x^5+c_0)*exp(-(4*log(x))))^(1/-2)]
```

3. Solve :

$$x^2y' - 2y = xe^{(4/x)}y^3$$

Input :

```
desolve(x*y'-2*y-x*exp(4/x)*y^3,y)
```

Output :

```
[((-2*ln(x)+c_0)*exp(-(4*(-(1/x)))))^(1/-2),

-(((-2*ln(x)+c_0)*exp(-(4*(-(1/x)))))^(1/-2))]
```

- Examples of first order homogeneous differential equations ($y' = F(y/x)$, the method of integration is to search $t = y/x$ instead of y).

1. Solve :

$$3x^3y' = y(3x^2 - y^2)$$

Input :

```
desolve(3*x^3*diff(y)=((3*x^2-y^2)*y),y)
```

Output :

```
[0,pnt[c_0*exp((3*1/2)/(` t`^2)),`
 t`*c_0*exp((3*1/2)/(` t`^2))]]
```

hence the solutions are $y = 0$ and the familiy of curves of parametric equation $x = c_0 \exp(3/(2t^2)), y = t * c_0 \exp(3/(2t^2))$ (the parameter is denoted by `` ` t` `` in the answer).

2. Solve :

$$xy' = y + \sqrt{x^2 + y^2}$$

Input :

```
desolve(x*y'=y+sqrt(x^2+y^2),y)
```

Output :

```
[(-i)*x,(i)*x,pnt[c_0/(sqrt(` t`^2+1)-` t`),(`
      t`*c_0)/(sqrt(` t`^2+1)-` t`)]]
```

hence the solutions are :

$$y = ix, y = -ix$$

and the family of curves of parametric equations

$$x = c_0/(\sqrt{t^2+1} - t), y = t * c_0/(\sqrt{t^2+1} - t)$$

(the parameter is denoted by `' t '` in the answer).

- Examples of first order differential equations with an integrating factor. By multiplying the equation by a function of x, y, it becomes a closed differential form.

  1. Solve :

     $$yy' + x$$

     Input :

     ```
     desolve(y*y'+x,y)
     ```

     Output :

     ```
     [sqrt(-2*c_0-x^2),-(sqrt(-2*c_0-x^2))]
     ```

     In this example, $xdx + ydy$ is closed, the integrating factor was 1.

  2. Solve :

     $$2xyy' + x^2 - y^2 + a^2 = 0$$

     Input :

     ```
     desolve(2*x*y*y'+x^2-y^2+a^2,y)
     ```

     Output :

     ```
     [sqrt(a^2-x^2-c_1*x),-(sqrt(a^2-x^2-c_1*x))]
     ```

     In this example, the integrating factor was $1/x^2$.

- Example of first order differential equations without x.
  Solve :

  $$(y + y')^4 + y' + 3y = 0$$

  This kind of equation cannot be solved directly by `Xcas`, we explain how to solve them with its help. The idea is to find a parametric representation of $F(u, v) = 0$ where the equation is $F(y, y') = 0$, Let $u = f(t), v = g(t)$ be such a parametrization of $F = 0$, then $y = f(t)$ and $dy/dx = y' = g(t)$. Hence

  $$dy/dt = f'(t) = y' * dx/dt = g(t) * dx/dt$$

  The solution is the curve of parametric equations $x(t), y(t) = f(t)$, where $x(t)$ is solution of the differential equation $g(t)dx = f'(t)dt$.
  Back to the example, we put $y + y' = t$, hence:

  $$y = -t - 8 * t^4, \quad y' = dy/dx = 3 * t + 8 * t^4 \quad dy/dt = -1 - 32 * t^3$$

  therefore

  $$(3 * t + 8 * t^4) * dx = (-1 - 32 * t^3)dt$$

  Input :

```
desolve((3*t+8*t^4)*diff(x(t),t)=(-1-32*t^3),x(t))
```

Output :

```
-11*1/9*ln(8*t^3+3)+1/-9*ln(t^3)+c_0
```

eventually the solution is the curve of parametric equation :

$$x(t) = -11*1/9*\ln(8*t^3+3)+1/-9*\ln(t^3)+c_0, \quad y(t) = -t-8*t^4$$

- Examples of first order Clairaut differential equations ($y = x * y' + f(y')$). The solutions are the lines D_m of equation $y = mx + f(m)$ where m is a real constant.

  1. Solve :
  $$xy' + y'^3 - y = 0$$
  Input :
  ```
  desolve(x*y'+y'^3-y),y)
  ```
  Output :
  ```
  c_0*x+c_0^3
  ```
  2. Solve :
  $$y - xy' - \sqrt{a^2 + b^2 * y'^2} = 0$$
  Input :
  ```
  desolve((y-x*y'-sqrt(a^2+b^2*y'^2),y)
  ```
  Output :
  ```
  c_0*x+sqrt(a^2+b^2*c_0^2)
  ```

5.55.2 Laplace transform and inverse Laplace transform : `laplace` `ilaplace`

`laplace` and `ilaplace` take one, two or three arguments : an expression and optionally the name(s) of the variable(s).
The expression is an expression of the current variable (here x) or an expression of the variable given as second argument.
`laplace` returns the Laplace transform of the expression given as argument and `ilaplace` the inverse Laplace transform of the expression given as argument. The result of `laplace` or `ilaplace` is expressed in terms of the variable given as third argument if supplied or second argument if supplied or x otherwise.

The Laplace transform (`laplace`) and inverse Laplace transform (`ilaplace`) are useful to solve linear differential equations with constant coefficients. For example :

$$y'' + p.y' + q.y = f(x)$$
$$y(0) = a, \ y'(0) = b$$

Denoting by $\mathcal{L}$ the Laplace transform, the following relations hold :

$$\begin{aligned} \mathcal{L}(y)(x) &= \int_0^{+\infty} e^{-xu} y(u) du \\ \mathcal{L}^{-1}(g)(x) &= \frac{1}{2i\pi} \int_C e^{zx} g(z) dz \end{aligned}$$

where C is a closed contour enclosing the poles of `g`.
Input :

```
laplace(sin(x))
```

The expression (here $\sin(x)$) is an expression of the current variable (here x) and the answer will also be an expression of the current variable x.
Output :

```
1/((-x)^2+1)
```

or :

```
laplace(sin(t),t)
```

here the variable name is t and this name is also used in the answer.
Output :

```
1/((-t)^2+1)
```

Or input :

```
laplace(sin(t),t,s)
```

here the variable name is t and the variable name of the answer is s.
Output:

```
1/((-s)^2+1)
```

The following properties hold :

$$\begin{aligned} \mathcal{L}(y')(x) &= -y(0) + x.\mathcal{L}(y)(x) \\ \mathcal{L}(y'')(x) &= -y'(0) + x.\mathcal{L}(y')(x) \\ &= -y'(0) - x.y(0) + x^2.\mathcal{L}(y)(x) \end{aligned}$$

If $y\prime\prime(x) + py\prime(x) + qy(x) \;=\; f(x)$, then :

$$\begin{aligned} \mathcal{L}(f)(x) &= \mathcal{L}(y'' + p.y' + q.y)(x) \\ &= -y'(0) - xy(0) + x^2\mathcal{L}(y)(x) - py(0) + px\mathcal{L}(y)(x)) + q\mathcal{L}(y)(x) \\ &= (x^2 + px + q)\mathcal{L}(y)(x) - y'(0) - (x + p)y(0) \end{aligned}$$

Therefore, if $a = y(0)$ and $b = y'(0)$, we have

$$\mathcal{L}(f)(x) = (x^2 + px + q).\mathcal{L}(y)(x) - (x + p)a - b$$

and the solution of the differential equation is :

$$y(x) = \mathcal{L}^{-1}((\mathcal{L}(f)(x) + (x + p)a + b)/(x^2 + px + q))$$

Example :
Solve :

$$y\prime\prime - 6y\prime + 9y \ = \ xe^{3.x}, \quad y(0) = c\_0, \quad y\prime(0) = c\_1$$

Here, $p = -6, \ q = 9$.
Input :

```
laplace(x*exp(3*x))
```

Output :

```
1/(x^ 2-6*x+9)
```

Input :

```
ilaplace((1/(x^2-6*x+9)+(x-6)*c_0+c_1)/(x^2-6*x+9))
```

Output :

```
(216*x^3-3888*x*c_0+1296*x*c_1+1296*c_0)*exp(3*x)/1296
```

After simplification and factorization (`factor` command) the solution y is :

```
(-18*c_0*x+6*c_0+x^3+6*x*c_1)*exp(3*x)/6
```

Note that this equation could be solved directly. Input :

```
desolve(y''-6*y'+9*y=x*exp(3*x),y)
```

Output :

```
exp(3*x)*(-18*c_0*x+6*c_0+x^3+6*x*c_1)/6
```

5.56 Other functions

5.56.1 Replace small values by 0: `epsilon2zero`

`epsilon2zero` takes as argument an expression of `x`.
`epsilon2zero` returns the expression where the values of modulus less than `epsilon` are replaced by zero. The expression is not evaluated.
The `epsilon` value is defined in the `cas` configuration (by default `epsilon=1e-10`).
Input :

```
epsilon2zero(1e-13+x)
```

Output (with `epsilon=1e-10`) :

```
0+x
```

Input :

```
epsilon2zero((1e-13+x)*100000)
```

Output (with `epsilon=1e-10`) :

```
(0+x)*100000
```

Input :

```
epsilon2zero(0.001+x)
```

Output (with `epsilon=0.0001`) :

```
0.001+x
```

5.56.2 List of variables : `lname indets`

`lname` (or `indets`) takes as argument an expression.
`lname` (or `indets`) returns the list of the symbolic variable names used in this expression.
Input :

```
lname(x*y*sin(x))
```

Output :

```
[x,y]
```

Input :

```
a:=2;assume(b>0);assume(c=3);

lname(a*x^2+b*x+c)
```

Output :

```
[x,b,c]
```

5.56.3 List of variables and of expressions : `lvar`

`lvar` takes as argument an expression.
`lvar` returns a list of variable names and non-rational expressions such that its argument is a rational fraction with respect to the variables and expressions of the list.
Input :

```
lvar(x*y*sin(x)^2)
```

Output :

```
[x,y,sin(x)]
```

Input :

```
lvar(x*y*sin(x)^2+ln(x)*cos(y))
```

Output :

```
[x,y,sin(x),ln(x),cos(y)]
```

Input :

```
lvar(y+x*sqrt(z)+y*sin(x))
```

Output :

```
[x,y,sqrt(z),sin(x)]
```

5.56.4 List of variables of an algebraic expressions: `algvar`

`algvar` takes as argument an expression.
`algvar` returns the list of the symbolic variable names used in this expression. The list is ordered by the algebraic extensions required to build the original expression.
Input :

```
algvar(y+x*sqrt(z))
```

Output :

```
[[y,x],[z]]
```

Input :

```
algvar(y*sqrt(x)*sqrt(z))
```

Output :

```
[[y],[z],[x]]
```

Input :

```
algvar(y*sqrt(x*z))
```

Output :

```
[[y],[x,z]]
```

Input :

```
algvar(y+x*sqrt(z)+y*sin(x))
```

Output :

```
[[x,y,sin(x)],[z]]
```

5.56.5 Test if a variable is in an expression : `has`

`has` takes as argument an expression and the name of a variable.
`has` returns `1` if this variable is in this expression, and else returns `0`.
Input :

```
has(x*y*sin(x),y)
```

Output :

```
1
```

Input :

```
has(x*y*sin(x),z)
```

Output :

```
0
```

5.56.6 Numeric evaluation : evalf

evalf takes as argument an expression or a matrix.
evalf returns the numeric value of this expression or of this matrix.
Input :

```
evalf(sqrt(2))
```

Output :

```
1.41421356237
```

Input :

```
evalf([[1,sqrt(2)],[0,1]])
```

Output :

```
[[1.0,1.41421356237],[0.0,1.0]]
```

5.56.7 Rational approximation : float2rational exact

float2rational (or exact) takes as argument an expression.
float2rational returns a rational approximation of all the floating point numbers r contained in this expression, such that $|r - \texttt{float2rational}(r)| < \epsilon$, where ϵ is defined by epsilon in the cas configuration (Cfg menu, or cas_setup command).
Input :

```
float2rational(1.5)
```

Output :

```
3/2
```

Input :

```
float2rational(1.414)
```

Output :

```
707/500
```

Input :

```
float2rational(0.156381102937*2)
```

Output :

```
5144/16447
```

Input :

```
float2rational(1.41421356237)
```

Output :

```
114243/80782
```

Input :

```
float2rational(1.41421356237^2)
```

Output :

```
2
```

Chapter 6

Graphs

Most graph instructions take expressions as arguments. A few exceptions (mostly Maple-compatibility instructions) also accept functions. Some optional arguments, like `color, thickness`, can be used as optional attributes in all graphic instructions. They are described below.

6.1 Graph and geometric objects attributes

There are two kinds of attributes: global attributes of a graphic scene and individual attributes.

6.1.1 Individual attributes

Graphic attributes are optional arguments of the form `display=value`, they must be given as the last argument of a graphic instruction. Attributes are ordered in several categories: color, point shape, point width, line style, line thickness, legend value, position and presence. In addition, surfaces may be filled or not, 3-d surfaces may be filled with a texture, 3-d objects may also have properties with respect to the light. Attributes of different categories may be added, e.g.
`plotfunc(`$x^2 + y^2$`,[x,y],display=red+line_width_3+filled)`

- Colors `display=` or `color=`
  - `black`, `white`, `red`, `blue`, `green`, `magenta`, `cyan`, `yellow`,
  - a numeric value between 0 and 255,
  - a numeric value between 256 and 256+7*16+14 for a color of the rainbow,
  - any other numeric value smaller than 65535, the rendering is not guaranteed to be portable.
- Point shapes `display=` one of the following value `rhombus_point plus_point square_point cross_point triangle_point star_point point_point invisible_point`
- Point width: `display=` one of the following value `point_width_n` where `n` is an integer between 1 and 7

- Line thickness: `thickness=n` or `display=line_width_n` where `n` is an integer between 1 and 7 or
- Line shape: `display=` one of the following values `dash_line solid_line dashdot_line dashdotdot_line cap_flat_line cap_square_line cap_round_line`
- Legend, value: `legend="legendname"`; position: `display=` one of `quandrant1 quadrant2 quadrant3 quadrant4` corresponding to the position of the legend of the object (using the trigonometric plane conventions). The legend is not displayed if the attribute `display=hidden_name` is added
- `display=filled` specifies that surfaces will be filled,
- `gl_texture="picture_filename"` is used to fill a surface with a texture. Cf. the interface manual for a more complete description and for `gl_material=` options.

Examples
Input :

```
polygon(-1,-i,1,2*i,legend="P")
```

Input :

```
point(1+i,legend="hello")
```

Input :

```
A:=point(1+i);B:=point(-1);display(D:=droite(A,B),hidden_name)
```

Input :

```
color(segment(0,1+i),red)
```

Input :

```
segment(0,1+i,color=red)
```

6.1.2 Global attributes

These attributes are shared by all objects of the same scene

- `title="titlename"` defines the title
- `labels=["xname","yname","zname"]`: names of the x, y, z axis
- `gl_x_axis_name="xname"`,`gl_y_axis_name="yname"`,`gl_z_axis_name=""`: individual definitions of the names of the x, y, z axis
- `legend=["xunit","yunit","zunit"]`: units for the x, y, z axis
- `gl_x_axis_unit="xunit"`,`gl_y_axis_unit="yunit"`,`gl_z_axis_unit=""`: individual definition of the units of the x, y, z axis

- `axes=true or false` show or hide axis
- `gl_texture="filename"`: background image
- `gl_x=xmin..xmax`, `gl_y=ymin..ymax`, `gl_z=zmin..zmax`: set the graphic configuration (do not use for interactive scenes)
- `gl_xtick=`, `gl_ytick=`, `gl_ztick=`: set the tick mark for the axis
- `gl_shownames=true or false`: show or hide objects names
- `gl_rotation=[x,y,z]`: defines the rotation axis for the animation rotation of 3-d scenes.
- `gl_quaternion=[x,y,z,t]`: defines the quaternion for the visualization in 3-d scenes (do not use for interactive scenes)
- a few other OpenGL light configuration options are available but not described here.

Examples
Input :

```
legend=["mn","kg"]
```

Input :

```
title="median_line";triangle(-1-i,1,1+i);median_line(-1-i,1,1+i);median_line(
```

Input :

```
labels=["u","v"];plotfunc(u+1,u)
```

6.2 Graph of a function : `plotfunc funcplot DrawFunc Graph`

6.2.1 2-d graph

`plotfunc(f(x),x)` draws the graph of $y = f(x)$ for x in the default interval, `plotfunc(f(x),x=a..b)` draws the graph of $y = f(x)$ for $a \leq x \leq b$. `plotfunc` accepts an optional `xstep=...` argument to specify the discretization step in x.
Input :

```
plotfunc(x^2-2)
```

or :

```
plotfunc(a^2-2,a=-1..2)
```

Output :

```
the graph of y=x^2-2
```

Input :

```
plotfunc(x^2-2,x,xstep=1)
```

Output :

```
a polygonal line which is a bad representation of
y=x^2-2
```

It is also possible to specify the number of points used for the representation of the function with `nstep=` instead of `xstep=`. For example, input :

```
plotfunc(x^2-2,x=-2..3,nstep=30)
```

6.2.2 3-d graph

`plotfunc` takes two main arguments : an expression of two variables or a list of several expressions of two variables and the list of these two variables, where each variable may be replaced by an equality variable=interval to specify the range for this variable (if not specified, default values are taken from the graph configuration). `plotfunc` accepts two optional arguments to specify the discretization step in x and in y by `xstep=...` and `ystep=...`. Alternatively one can specify the number of points used for the representation of the function with `nstep=` (instead of `xstep` and `ystep`).
`plotfunc` draws the surface(s) defined by $z =$ the first argument.
Input :

```
plotfunc( x^2+y^2,[x,y])
```

Output :

```
A 3D graph of z=x^2+y^2
```

Input :

```
plotfunc(x*y,[x,y])
```

Output :

```
The surface z=x*y, default ranges
```

Input :

```
plotfunc([x*y-10,x*y,x*y+10],[x,y])
```

Output :

```
The surfaces z=x*y-10, z=x*y and z=x*y+10
```

Input :

```
plotfunc(x*sin(y),[x=0..2,y=-pi..pi])
```

Output :

The surface $z = x * y$ for the specified ranges

Now an example where we specify the x and y discretization step with `xstep` and `ystep`.
Input :

```
plotfunc(x*sin(y),[x=0..2,y=-pi..pi],xstep=1,ystep=0.5)
```

Output :

A portion of surface $z = x * y$

Alternatively we may specify the number of points used for the representation of the function with `nstep` instead of `xstep` and `ystep`.
Input :

```
plotfunc(x*sin(y),[x=0..2,y=-pi..pi],nstep=300)
```

Output :

A portion of surface $z = x * y$

Remarks

- Like any 3-d scene, the viewpoint may be modified by rotation around the `x` axis, the `y` axis or the `z` axis, either by dragging the mouse inside the graphic window (push the mouse outside the parallelepiped used for the representation), or with the shortcuts `x`, `X`, `y`, `Y`, `z` and `Z`.
- If you want to print a graph or get a LaTeX translation, use the graph menu `Menu►print►Print(with Latex)`

6.2.3 3-d graph with rainbow colors

`plotfunc` represents a pure imaginary expression `i*E` of two variables with a rainbow color depending on the value of `z=E`. This gives an easy way to find points having the same third coordinate.
The first arguments of `plotfunc` must be `i*E` instead of `E`, the remaining arguments are the same as for a real 3-d graph (cf 6.2.2) Input :

```
plotfunc(i*x*sin(y),[x=0..2,y=-pi..pi])
```

Output :

A piece of the surface $z = x * \sin(y)$ with rainbow colors

Remark
If you want the graphic in LaTeX, you have to use :
`Menu►print►Print(with Latex)`.

6.2.4 4-d graph.

`plotfunc` represents a complex expression `E` (such that `re(E)` is not identically 0 on the discretization mesh) by the surface `z=abs(E)` where `arg(E)` defines the color from the rainbow. This gives an easy way to see the points having the same argument. Note that if `re(E)==0` on the discretization mesh, it is the surface `z=E/i` that is represented with rainbow colors (cf 6.2.3).
The first argument of `plotfunc` is `E`, the remaining arguments are the same as for a real 3-d graph (cf 6.2.2).
Input :

```
plotfunc((x+i*y)^2,[x,y])
```

Output :

```
A graph 3D of z=abs((x+i*y)^2 with the same color for
points having the same argument
```

Input :

```
plotfunc((x+i*y)^2x,[x,y], display=filled)
```

Output :

```
The same surface but filled
```

We may specify the range of variation of x and y and the number of discretization points.
Input :

```
plotfunc((x+i*y)^2,[x=-1..1,y=-2..2],
         nstep=900,display=filled)
```

Output :

The specified part of the surface with x between -1 and 1, y between -2 and 2 and with 900 points

6.3 2d graph for Maple compatibility : `plot`

`plot(f(x),x)` draws the graph of $y = f(x)$. The second argument may specify the range of values `x=xmin..xmax`. One can also plot a function instead of an expression using the syntax `plot(f,xmin..xmax)`. `plot` accepts an optional argument to specify the step used in x for the discretization with `xstep=` or the number of points of the discretization with `nstep=`.
Input :

```
plot(x^2-2,x)
```

Output :

```
the graph of y=x^2-2
```

Input :

```
plot(x^2-2,xstep=1)
```

or :

```
plot(x^2-2,x,xstep=1)
```

Output :

```
a polygonal line which is a bad representation of
y=x^2-2
```

Input!

```
plot(x^2-2,x=-2..3,nstep=30)
```

6.4 3d surfaces for Maple compatibility `plot3d`

`plot3d` takes three arguments : a function of two variables or an expression of two variables or a list of three functions of two variables or a list of three expressions of two variables and the names of these two variables with an optional range (for expressions) or the ranges (for functions).
`plot3d(f(x,y),x,y)` (resp. `plot3d([f(u,v),g(u,v),h(u,v)],u,v)`) draws the surface $z = f(x, y)$ (resp. $x = f(u, v), y = g(u, v), z = h(u, v)$). The syntax `plot3d(f(x,y),x=x0..x1,y=y0..y1)` or `plot3d(f,x0..x1,y0..y1)` specifies which part of surface will be computed (otherwise default values are taken from the graph configuration).
Input :

```
plot3d(x*y,x,y)
```

Output :

The surface $z = x * y$

Input :

```
plot3d([v*cos(u),v*sin(u),v],u,v)
```

Output :

The cone $x = v * \cos(u), y = v * \sin(u), z = v$

Input :

```
plot3d([v*cos(u),v*sin(u),v],u=0..pi,v=0..3)
```

Output :

A portion of the cone $x = v * \cos(u), y = v * \sin(u), z = v$

6.5 Graph of a line and tangent to a graph

6.5.1 Draw a line : `line`

See also : ?? and **??** for line usage in geometry and see **??** and **??** for axis.
`line` takes as argument cartesian equation(s) :

- in 2D: one line equation,
- in 3D: two plane equations.

`line` defines and draws the corresponding line.
Input :

```
line(2*y+x-1=0)
```

Output :

```
the line 2*y+x-1=0
```

Input :

```
line(y=1)
```

Output :

```
the horizontal line y=1
```

Input :

```
line(x=1)
```

Output :

```
the vertical line x=1
```

Input :

```
line(x+2*y+z-1=0,z=2)
```

Output :

```
the line x+2*y+1=0 in the plane z=2
```

Input :

```
line(y=1,x=1)
```

Output :

```
the vertical line crossing through (1,1,0)
```

Remark
`line` defines an oriented line :

- when the 2D line is given by an equation, it is rewritten as "left_member-right_member=`ax+by+c=0`", this determines its normal vector `[a,b]` and the orientation is given by the vector `[b,-a]`) (or its orientation is defined by the 3D cross product of its normal vectors (with third coordinate 0) and the vector [0,0,1]).
  For example `line(y=2*x)` defines the line `-2x+y=0` with as direction the vector `[1,2]` (or `cross([-2,1,0],[0,0,1])=[1,2,0]`).
- when the 3D line is given by two plane equations, its direction is defined by the cross product of the normals to the planes (where the plane equation is rewritten as "left_member-right_member=`ax+by+cz+d=0`", so that the normal is `[a,b,c]`).
  For example the `line(x=y,y=z)` is the line `x-y=0,y-z=0` and its direction is :
  `cross([1,-1,0],[0,1,-1])=[1,1,1].`

6.5.2 Draw an 2D horizontal line : `LineHorz`

`LineHorz` takes as argument an expression a.
`LineHorz` draws the horizontal line $y = a$.
Input :

```
LineHorz(1)
```

Output :

```
the line y=1
```

6.5.3 Draw a 2D vertical line : `LineVert`

`LineVert` takes as argument an expression a.
`LineVert` draws the vertical line $x = a$.
Input :

```
LineVert(1)
```

Output :

```
the line x=1
```

6.5.4 Tangent to a 2D graph : `LineTan`

`LineTan` takes two arguments : an expression E_x of the variable x and a value $x0$ of x.
`LineTan` draws the tangent at $x = x0$ to the graph of $y = E_x$.
Input :

```
LineTan(ln(x),1)
```

Output :

```
the line y=x-1
```

Input :

```
equation(LineTan(ln(x),1))
```

Output :

```
y=(x-1)
```

6.5.5 Tangent to a 2D graph : `tangent`

See also : ?? for plane geometry and **??** for 3D geometry.
`tangent` takes two arguments : a geometric object and a point `A`.
`tangent` draws tangent(s) to this geometric object crossing through `A`. If the geometric object is the graph `G` of a 2D function, the second argument is either, a real number `x0`, or a point `A` on `G`. In that case `tangent` draws a tangent to this graph `G` crossing through the point `A` or through the point of abscissa `x0`.
For example, define the function `g`

```
g(x):=x^2
```

then the graph `G={(x,y)`$\in \mathbb{R}^2$`, y=g(x)}` of g and a point A on the graph G:

```
G:=plotfunc(g(x),x);
A:=point(1.2,g(1.2));
```

If we want to draw the tangent at the point `A` to the graph `G`, we will input:

```
T:=tangent(G, A)
```

or :

```
T:=tangent(G, 1.2)
```

For the equation of the tangent line, input :

```
equation(T)
```

6.5.6 Intersection of a 2D graph with the axis

- The ordinate of the intersection of the graph of f with the y-axis is returned by :

  ```
  f(0)
  ```

  indeed the point of coordinates $(0, f(0))$ is the intersection point of the graph of f with the y-axis,

- Finding the intersection of the graph of f with the x-axis requires solving the equation $f(x) = 0$.
  If the equation is polynomial-like, `solve` will find the exact values of the abscissa of these points. Input:

  ```
  solve(f(x),x)
  ```

  Otherwise, we can find numeric approximations of these abscissa. First look at the graph for an initial guess or a range with an intersection and refine with `fsolve`.

6.6 Graph of inequalities with 2 variables : plotinequation inequationplot

`plotinequation([f1(x,y)<a1,...fk(x,y)<ak],[x=x1..x2,y=y1..y2])` draws the points of the plane whose coordinates satisfy the inequalities of 2 variables :

$$\begin{cases} f1(x,y) & < & a1 \\ & ... & \\ fk(x,y) & < & ak \end{cases}, \quad x1 \le x \le x2, y1 \le y \le y2$$

Input :

```
plotinequation(x^2-y^2<3,
[x=-2..2,y=-2..2],xstep=0.1,ystep=0.1)
```

Output :

```
the filled portion enclosing the origin and limited by
the hyperbola x^2-y^2=3
```

Input :

```
plotinequation([x+y>3,x^2<y],
[x-2..2,y=-1..10],xstep=0.2,ystep=0.2)
```

Output :

```
the filled portion of the plane defined by
-2<x<2,y<10,x+y>3,y>x^2
```

Note that if the ranges for x and y are not specified, Xcas takes the default values of X-,X+,Y-,Y+ defined in the general graphic configuration (Cfg▶Graphic configuration).

6.7 Graph of the area below a curve : plotarea areaplot

- With two arguments, plotarea shades the area below a curve.
  plotarea(f(x),x=a..b) draws the area below the curve $y = f(x)$ for $a < x < b$, i.e. the portion of the plane defined by the inequalities $a < x < b$ and $0 < y < f(x)$ or $0 > y > f(x)$ according to the sign of $f(x)$.
  Input :

  ```
  plotarea(sin(x),x=0..2*pi)
  ```

  Output :

  ```
  the portion of plane locates in the two arches of
  sin(x)
  ```

- With four arguments, `plotarea` represents a numeric approximation of the area below a curve, according to a quadrature method from the following list: `trapezoid,rectangle_left,rectangle_right,middle_point`. For example `plotarea(f(x),x=a..b,n,trapezoid)` draws the area of n trapezoids : the third argument is an integer n, and the fourth argument is the name of the numeric method of integration when $[a, b]$ is cut into n equal parts.
  Input :

  ```
  plotarea((x^2,x=0..1,5,trapezoid)
  ```

  If you want to display the graph of the curve in contrast (e.g. in bold red), input :

  ```
  plotarea(x^2,x=0..1,5,trapezoid);
  plot(x^2,x=0..1,display=red+line_width_3)
  ```

  Output :

  ```
  the 5 trapezoids used in the trapezoid method to
  approach the integral
  ```

  Input :

  ```
  plotarea((x^2,x=0..1,5,middle_point)
  ```

  Or with the graph of the curve in bold red, input :

  ```
  plotarea(x^2,x=0..1,5,middle_point);
  plot(x^2,x=0..1,display=red+line_width_3)
  ```

  Output :

  ```
  the 5 rectangles used in the middle_point method
  to approach the integral
  ```

6.8 Contour lines: `plotcontour contourplot DrwCtour`

`plotcontour(f(x,y),[x,y])` (or `DrwCtour(f(x,y),[x,y])` or `contourplot(f(x,y),[x,y])`) draws the contour lines of the surface defined by $z = f(x, y)$ for $z = -10$, $z = -8$, .., $z = 0$, $z = 2$, .., $z = 10$. You may specify the desired contour lines by a list of values of z given as third argument.
Input :

```
plotcontour(x^2+y^2,[x=-3..3,y=-3..3],[1,2,3],
    display=[green,red,black]+[filled$3])
```

Output :

```
the graph of the three ellipses x^2-y^2=n for n=1,2,3;
 the zones between these ellipses are filled with the
              color green,red or black
```

Input :

```
plotcontour(x^2-y^2,[x,y])
```

Output :

```
the graph of 11 hyperbolas x^2-y^2=n for n=-10,-8,..10
```

If you want to draw the surface in 3-d representation, input `plotfunc(f(x,y),[x,y])`, see 6.2.2):

```
plotfunc( x^2-y^2,[x,y])
```

Output :

```
A 3D representation of z=x^2+y^2
```

6.9 2-d graph of a 2-d function with colors : `plotdensity densityplot`

`plotdensity(f(x,y),[x,y])` or `densityplot(f(x,y),[x,y])` draws the graph of $z = f(x, y)$ in the plane where the values of z are represented by the rainbow colors. The optional argument `z=zmin..zmax` specifies the range of z corresponding to the full rainbow, if it is not specified, it is deduced from the minimum and maximum value of f on the discretization. The discretization may be specified by optional `xstep=...` and `ystep=...` or `nstep=...` arguments.
Input :

```
plotdensity(x^2-y^2,[x=-2..2,y=-2..2],
         xstep=0.1,ystep=0.1)
```

Output :

```
A 2D graph where each hyperbola defined by x^2-y^2=z
            has a color from the rainbow
```

Remark : A rectangle representing the scale of colors is displayed below the graph.

6.10 Implicit graph: `plotimplicit implicitplot`

`plotimplicit` or `implicitplot` draws curves or surfaces defined by an implicit expression or equation. If the option `unfactored` is given as last argument, the original expression is taken unmodified. Otherwise, the expression is normalized, then replaced by the factorization of the numerator of its normalization.

Each factor of the expression corresponds to a component of the implicit curve or surface. For each factor, Xcas tests if it is of total degree less or equal to 2, in that case `conic` or `quadric` is called. Otherwise the numeric implicit solver is called.

Optional step and ranges arguments may be passed to the numeric implicit solver, note that they are dismissed for each component that is a conic or a quadric.

6.10.1 2D implicit curve

- `plotimplicit(f(x,y),x,y)` draws the graphic representation of the curve defined by the implicit equation $f(x, y) = 0$ when x (resp. y) is in `WX-, WX+` (resp. in `WY-, WY+`) defined by `cfg`,
- `plotimplicit(f(x,y),x=0..1,y=-1..1)` draws the graphic representation of the curve defined by the implicit equation $f(x, y) = 0$ when $0 \leq x \leq 1$ and $-1 \leq y \leq 1$

It is possible to add two arguments to specify the discretization steps for x and y with `xstep=...` and `ystep=...`.
Input :

```
plotimplicit(x^2+y^2-1,x,y)
```

or :

```
plotimplicit(x^2+y^2-1,x,y,unfactored)
```

Output :

```
The unit circle
```

Input :

```
plotimplicit(x^2+y^2-1,x,y,xstep=0.2,ystep=0.3)
```

or :

```
plotimplicit(x^2+y^2-1,[x,y],xstep=0.2,ystep=0.3)
```

or :

```
plotimplicit(x^2+y^2-1,[x,y],
xstep=0.2,ystep=0.3,unfactored)
```

Output :

```
The unit circle
```

Input :

```
plotimplicit(x^2+y^2-1,x=-2..2,y=-2..2,
xstep=0.2,ystep=0.3)
```

Output :

```
The unit circle
```

6.10.2 3D implicit surface

- `plotimplicit(f(x,y,z),x,y,z)` draws the graphic representation of the surface defined by the implicit equation $f(x, y, z) = 0$,
- `plotimplicit(f(x,y,z),x=0..1,y=-1..1,z=-1..1)` draws the surface defined by the implicit equation $f(x, y, z) = 0$, where $0 \leq x \leq 1$, $-1 \leq y \leq 1$ and $-1 \leq z \leq 1$.

It is possible to add three arguments to specify the discretization steps used for x, y and z with `xstep=...`, `ystep=...` and `zstep=...`.
Input :

```
plotimplicit(x^2+y^2+z^2-1,x,y,z,
  xstep=0.2,ystep=0.1,zstep=0.3)
```

Input :

```
plotimplicit(x^2+y^2+z^2-1,x,y,z,
xstep=0.2,ystep=0.1,zstep=0.3,unfactored)
```

Output :

```
The unit sphere
```

Input :

```
plotimplicit(x^2+y^2+z^2-1,x=-1..1,y=-1..1,z=-1..1)
```

Output :

```
The unit sphere
```

6.10.3 Implicit differentiation : `implicitdiff`

`implicitdiff` is called with one of the following three sets of parameters :

1. `expr`, `constr`, `depvars`, `diffvars`
2. `constr`, `[depvars]`, `y`, `diffvars`
3. `expr`, `constr`, `vars`, `order_size=k`, `[pt]`

Details on parameters :

- `expr` : differentiable expression $f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$
- `constr` : (list of) equality constraint(s) $g_i(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$ or vanishing expression(s) g_i, where $i = 1, 2, \ldots, m$
- `depvars` : (list of) dependent variable(s) $y_1, y_2, \ldots, y_m$, each of which may be entered as a symbol, e.g. `yi`, or a function of independent variable(s), e.g. `yi(x1,x2,..,xn)`
- `diffvars` : sequence of variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ with respect to which is `expr` differentiated

- `vars` : independent and dependent variables entered as symbols in single list such that dependent variables come last, e.g. `[x1,..,xn,y1,..,ym]`
- `y` : (list of) dependent variable(s) $y_{j_1}, y_{j_2}, \ldots, y_{j_l}$ that need to be differentiated

Dependent variables $y_1, y_2, \ldots, y_m$ are implicitly defined with m constraints in `constr`. By implicit function theorem, the Jacobian matrix of $\mathbf{g} = (g_1, g_2, \ldots, g_m)$ has to be full rank.

When calling `implicitdiff`, first two sets of parameters are used when specific partial derivative is needed. In the first case, `expr` is differentiated with respect to `diffvars`.
Input :

```
implicitdiff(x*y,-2x^3+15x^2*y+11y^3-24y=0,y(x),x)
```

Output :

```
(2*x^3-5*x^2*y+11*y^3-8*y)/(5*x^2+11*y^2-8)
```

In the second case (elements of) `y` is differentiated. If `y` is a list of symbols, a list containing their derivatives will be returned. The following examples compute $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$.
Input :

```
implicitdiff(x^2*y+y^2=1,y,x)
```

Output :

```
-2*x*y/(x^2+2*y)
```

Input :

```
implicitdiff([x^2+y=z,x+y*z=1],[y(x),z(x)],y,x)
```

Output :

```
(-2*x*y-1)/(y+z)
```

In the next example, $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ and $\frac{\mathrm{d}\,z}{\mathrm{d}\,x}$ are computed.
Input :

```
implicitdiff([-2x*z+y^2=1,x^2-exp(x*z)=y],
[y(x),z(x)],[y,z],x)
```

Output :

```
[2*x/(y*exp(x*z)+1),
(2*x*y-y*z*exp(x*z)-z)/(x*y*exp(x*z)+x)]
```

For the third case of input syntax, all partial derivatives of order equal to `order_size`, i.e. k, are computed. If $k = 1$ they are returned in a single list, which represents the gradient of `expr` with respect to independent variables. For $k = 2$ the corresponding hessian matrix is returned. When $k > 2$, a table with keys in form `[k1,k2,..,kn]`, where $\sum_{i=1}^{n} k_i = k$, is returned. Such key corresponds to

$$\frac{\partial^k f}{\partial x_1^{k_1}\,\partial x_2^{k_2}\,\cdots\,\partial x_n^{k_n}}.$$

Input :

```
f:=x*y*z; g:=-2x^3+15x^2*y+11y^3-24y=0;
implicitdiff(f,g,[x,z,y],order_size=1)
```

Output :

```
[(2*x^3*z-5*x^2*y*z+11*y^3*z-8*y*z)/(5*x^2+11*y^2-8),
x*y]
```

Input :

```
implicitdiff(f,g,order_size=2,[1,-1,0])
```

Output :

```
[[64/9,-2/3],[-2/3,0]]
```

In the next example, the value of $\frac{\partial^4 f}{\partial x^4}$ is computed at point $(x = 0, y = 0, z)$.
Input :

```
pd:=implicitdiff(f,g,[x,z,y],order_size=4,[0,z,0]);
pd[4,0]
```

Output :

```
-2*z
```

6.11 Parametric curves and surfaces : plotparam paramplot DrawParm

6.11.1 2D parametric curve

`plotparam([f(t),g(t)],t)` or `plotparam(f(t)+i*g(t),t)` (resp. `plotparam(f(t)+i*g(t),t=t1..t2))` draws the parametric representation of the curve defined by $x = f(t), y = g(t)$ with the default range of values of t (resp. for $t1 \leq t \leq t2$).
The default range of values is taken as specified in the graphic configuration (`t-` and `t+`, cf. **??**). `plotparam` accepts an optional argument to specify the discretization step for t with `tstep=`.
Input :

```
plotparam(cos(x)+i*sin(x),x)
```

or :

```
plotparam([cos(x),sin(x)],x)
```

Output :

```
The unit circle
```

If in the graphic configuration `t` goes from -4 to 1, input :

```
plotparam(sin(t)+i*cos(t))
```

or :

```
plotparam(sin(t)+i*cos(t),t=-4..1)
```

or :

```
plotparam(sin(x)+i*cos(x),x=-4..1)
```

Output :

the arc (sin(-4)+i*cos(-4),sin(1)+i*cos(1)) of the unit circle

If in the graphic configuration t goes from -4 to 1, input :

```
plotparam(sin(t)+i*cos(t),t,tstep=0.5)
```

or :

```
plotparam(sin(t)+i*cos(t),t=-4..1,tstep=0.5)
```

Output :

A polygon approaching the arc (sin(-4)+i*cos(-4),sin(1)+i*cos(1)) of the unit circle

6.11.2 3D parametric surface : plotparam paramplot DrawParm

plotparam takes two main arguments, a list of three expressions of two variables and the list of these variable names where each variable name may be replaced by variable=interval to specify the range of the parameters. It accepts an optional argument to specify the discretization steps of the parameters u and v with ustep=... and vstep=....
plotparam([f(u,v),g(u,v),h(u,v)],[u,v]) draws the surface defined by the first argument : $x = f(u,v), y = g(u,v), z = h(u,v)$, where u and v ranges default to the graphic configuration.
Input :

```
plotparam([v*cos(u),v*sin(u),v],[u,v])
```

Output :

The cone $x = v * \cos(u), y = v * \sin(u), z = v$

To specify the range of each parameters, replace each variable by an equation variable=range, like this:

```
plotparam([v*cos(u),v*sin(u),v],[u=0..pi,v=0..3])
```

Output :

A portion of the cone $x = v * \cos(u), y = v * \sin(u), z = v$

Input :

```
plotparam([v*cos(u),v*sin(u),v],[u=0..pi,v=0..3],ustep=0.5,vstep=0.5)
```

Output :

A portion of the cone $x = v * \cos(u), y = v * \sin(u), z = v$

6.12 Curve defined in polar coordinates : plotpolar polarplot DrawPol courbe_polaire

Let E_t be an expression depending on the variable t.
plotpolar(E_t,t) draws the polar representation of the curve defined by $\rho = E_t$ for $\theta = t$, that is in cartesian coordinates the curve $(E_t \cos(t), E_t \sin(t))$. The range of the parameter may be specified by replacing the second argument by t=tmin..tmax. The discretization parameter may be specified by an optional tstep=... argument.
Input

```
plotpolar(t,t)
```

Output :

```
The spiral ρ=t is plotted
```

Input

```
plotpolar(t,t,tstep=1)
```

or :

```
plotpolar(t,t=0..10,tstep=1)
```

Output :

```
A polygon line approaching the spiral ρ=t is plotted
```

6.13 Graph of a recurrent sequence : plotseq seqplot graphe_suite

Let $f(x)$ be an expression depending on the variable x (resp. $f(t)$ an expression depending on the variable t).
plotseq($f(x)$,a,n) (resp. plotseq($f(t)$,t=a,n)) draws the line $y = x$, the graph of $y = f(x)$ (resp. $y = f(t)$) and the n first terms of the recurrent sequence defined by : $u_0 = a, \;\; u_n = f(u_{n-1})$. The a value may be replaced by a list of 3 elements, $[a, x_-, x_+]$ where $x_-..x_+$ will be passed as x range for the graph computation.
Input :

```
plotseq(sqrt(1+x),x=[3,0,5],5)
```

Output :

```
the graph of y=sqrt(1+x), of y=x and of the 5 first
terms of the sequence u_0=3 and u_n=sqrt(1+u_(n-1))
```

6.14 Tangent field : `plotfield fieldplot`

- Let $f(t, y)$ be an expression depending on two variables t and y, then :

  `plotfield(f(t,y),[t,y])`

  draws the tangent field of the differential equation $y' = f(t, y)$ where y is a real variable and where t is the abscissa,

- Let V be a vector of two expressions depending on 2 variables x, y but independent of the time t, then

  `plotfield(V,[x,y])`

  draws the vector field V,

- The range of values of t, y or of x, y can be specified with `t=tmin..tmax`, `x=xmin..xmax`, `y=ymin..ymax` in place of the variable name.

- The discretization may be specified with optional arguments `xstep=...`, `ystep=...`.

Input :

```
plotfield(4*sin(t*y),[t=0..2,y=-3..7])
```

Output :

```
Segments with slope 4*sin(t*y), representing tangents,
          are plotting in different points
```

With two variables x, y, input :

```
plotfield(5*[-y,x],[x=-1..1,y=-1..1])
```

6.15 Plotting a solution of a differential equation : `plotode odeplot`

Let $f(t, y)$ be an expression depending on two variables t and y.

- `plotode(`$f(t, y)$`,[t,y],[t0,y0])` draws the solution of the differential equation $y' = f(t, y)$ crossing through the point `(t0,y0)` (i.e. such that $y(t_0) = y_0$)

- By default, t goes in both directions. The range of value of t may be specified by the optional argument `t=tmin..tmax`.

- We can also represent, in the space or in the plane, the solution of a differential equation $y' = f(t, y)$ where $y = (X, Y)$ is a vector of size 2. Just replace y by the variable names X, Y and the initial value y_0 by the two initial values of the variables at time t_0.

Input :

```
plotode(sin(t*y),[t,y],[0,1])
```

Output :

The graph of the solution of y'=sin(t,y) crossing through the point (0,1)

Input :

```
S:=odeplot([h-0.3*h*p, 0.3*h*p-p],
      [t,h,p],[0,0.3,0.7])
```

Output, the graph in the space of the solution of :

$$[h, p]' = [h - 0.3h * p, 0.3h * p - p] \quad [h, p](0) = [0.3, 0.7]$$

To have a 2-d graph (in the plane), use the option `plane`

```
S:=odeplot([h-0.3*h*p, 0.3*h*p-p],
    [t,h,p],[0,0.3,0.7],plane)
```

To compute the values of the solution, see the section **??**.

6.16 Interactive plotting of solutions of a differential equation : `interactive_plotode` `interactive_odeplot`

Let $f(t, y)$ be an expression depending on two variables t and y.
`interactive_plotode(f(t,y),[t,y])` draws the tangent field of the differential equation $y' = f(t, y)$ in a new window. In this window, one can click on a point to get the plot of the solution of $y' = f(t, y)$ crossing through this point. You can further click to display several solutions. To stop press the `Esc` key.
Input :

```
interactive_plotode(sin(t*y),[t,y])
```

Output :

```
The tangent field is plotted with the solutions of
y'=sin(t,y) crossing through the points defined by
                  mouse clicks
```

6.17 Animated graphs (2D, 3D or "4D")

`Xcas` can display animated 2D, 3D or "4D" graphs. This is done first by computing a sequence of graphic objects, then after completion, by displaying the sequence in a loop.

- To stop or start again the animation, click on the button ▶| (at the left of `Menu`).
- The display time of each graphic object is specified in `animate` of the graph configuration (`cfg` button). Put a small time, to have a fast animation.
- If `animate` is `0`, the animation is frozen, you can move in the sequence of objects one by one by clicking on the mouse in the graphic scene.

6.17.1 Animation of a 2D graph : animate

animate can create a 2-d animation with graphs of functions depending on a parameter. The parameter is specified as the third argument of animate, the number of pictures as fourth argument with frames=number, the remaining arguments are the same as those of the plot command, see section 6.3, p. 402.
Input :

```
animate(sin(a*x),x=-pi..pi,a=-2..2,frames=10,color=red)
```

Output :

a sequence of graphic representations of y=sin(ax) for 11 values of a between -2 and 2

6.17.2 Animation of a 3D graph : animate3d

animate3d can create a 3-d animation with function graphs depending on a parameter. The parameter is specified as the third argument of animate3d, the number of pictures as fourth argument with frames=number, the remaining arguments are the same as those of the plotfunc command, see section 6.2.2, p. 400.
Input :

```
animate3d(x^2+a*y^2,[x=-2..2,y=-2..2],a=-2..2,
          frames=10,display=red+filled)
```

Output :

a sequence of graphic representations of z=x^2+a*y^2 for 11 values of a between -2 and 2

6.17.3 Animation of a sequence of graphic objects : animation

animation animates the representation of a sequence of graphic objects with a given display time. The sequence of objects depends most of the time on a parameter and is defined using the seq command but it is not mandatory.
animation takes as argument the sequence of graphic objects.
To define a sequence of graphic objects with seq, enter the definition of the graphic object (depending on the parameter), the parameter name, its minimum value, its maximum value maximum and optionally a step value.
Input :

```
animation(seq(plotfunc(cos(a*x),x),a,0,10))
```

Output :

The sequence of the curves defined by $y = \cos(ax)$, for $a = 0, 1, 2..10$

Input :

```
animation(seq(plotfunc(cos(a*x),x),a,0,10,0.5))
```

or :

```
animation(seq(plotfunc(cos(a*x),x),a=0..10,0.5))
```

Output :

The sequence of the curves defined by $y = \cos(ax)$, for $a = 0, 0.5, 1, 1.5..10$

Input :

```
animation(seq(plotfunc([cos(a*x),sin(a*x)],x=0..2*pi/a),
                         a,1,10))
```

Output :

The sequence of two curves defined by $y = \cos(ax)$ and $y = \sin(ax)$, for $a = 1..10$ and for $x = 0..2\pi/a$

Input :

```
animation(seq(plotparam([cos(a*t),sin(a*t)],
              t=0..2*pi),a,1,10))
```

Output :

The sequence of the parametric curves defined by $x = \cos(at)$ and $y = \sin(at)$, for $a = 1..10$ and for $t = 0..2\pi$

Input :

```
animation(seq(plotparam([sin(t),sin(a*t)],
      t,0,2*pi,tstep=0.01),a,1,10))
```

Output :

The sequence of the parametric curves defined by $x = \sin(t), y = \sin(at)$, for $a = 0..10$ and $t = 0..2\pi$

Input :

```
animation(seq(plotpolar(1-a*0.01*t^2,
    t,0,5*pi,tstep=0.01),a,1,10))
```

Output :

The sequence of the polar curves defined by $\rho = 1 - a * 0.01 * t^2$, for $a = 0..10$ and $t = 0..5\pi$

Input :

```
plotfield(sin(x*y),[x,y]);
animation(seq(plotode(sin(x*y),[x,y],[0,a]),a,-4,4,0.5))
```

Output :

The tangent field of y'=sin(xy) and the sequence of the integral curves crossing through the point $(0, a)$ for a=-4,-3.5...3.5,4

Input :

```
animation(seq(display(square(0,1+i*a),filled),a,-5,5))
```

Output :

The sequence of the squares defined by the points 0 and 1+i*a for $a = -5..5$

Input :

```
animation(seq(droite([0,0,0],[1,1,a]),a,-5,5))
```

Output :

The sequence of the lines defined by the points [0,0,0] and [1,1,a] for $a = -5..5$

Input :

```
animation(seq(plotfunc(x^2-y^a,[x,y]),a=1..3))
```

Output :

The sequence of the "3D" surface defined by $x^2 - y^a$, for $a = 1..3$ with rainbow colors

Input :

```
animation(seq(plotfunc((x+i*y)^a,[x,y],
display=filled),a=1..10)
```

Output :

The sequence of the "4D" surfaces defined by $(x + i * y)^a$, for $a = 0..10$ with rainbow colors

Remark We may also define the sequence with a program, for example if we want to draw the segments of length $1, \sqrt{2}...\sqrt{20}$ constructed with a right triangle of side 1 and the previous segment (note that there is a `c:=evalf(..)` statement to force approx. evaluation otherwise the computing time would be too long) :

```
seg(n):={
 local a,b,c,j,aa,bb,L;
 a:=1;
 b:=1;
 L:=[point(1)];
 for(j:=1;j<=n;j++){
  L:=append(L,point(a+i*b));
```

```
  c:=evalf(sqrt(a^2+b^2));
  aa:=a;
  bb:=b;
  a:=aa-bb/c;
  b:=bb+aa/c;
 }
 L;
}
```

Then input :

`animation(seg(20))`

We see, each point, one to one with a display time that depends of the `animate` value in `cfg`.
or :

`L:=seg(20); s:=segment(0,L[k])$(k=0..20)`

We see 21 segments.
Then, input :

`animation(s)`

We see, each segment, one to one with a display time that depends of the `animate` value in `cfg`.

Chapter 7

Numerical computations

Real numbers may have an exact representation (e.g. rationals, symbolic expressions involving square roots or constants like π, ...) or approximate representation, which means that the real is represented by a rational (with a denominator that is a power of the basis of the representation) close to the real. Inside `Xcas`, the standard scientific notation is used for approximate representation, that is a mantissa (with a point as decimal separator) optionally followed by the letter `e` and an integer exponent.

Note that the real number 10^{-4} is an exact number but $1e-4$ is an approximate representation of this number.

7.1 Floating point representation.

In this section, we explain how real numbers are represented.

7.1.1 Digits

The `Digits` variable is used to control how real numbers are represented and also how they are displayed. When the specified number of digits is less or equal to 14 (for example `Digits:=14`), then hardware floating point numbers are used and they are displayed using the specified number of digits. When `Digits` is larger than 14, Xcas uses the MPFR library, the representation is similar to hardware floats (cf. infra) but the number of bits of the mantissa is not fixed and the range of exponents is much larger. More precisely, the number of bits of the mantissa of a created MPFR float is `ceil(Digits*log(10)/log(2))`.

Note that if you change the value of `Digits`, this will affect the creation of new real numbers compiled from command lines or programs or by instructions like `approx`, but it will not affect existing real numbers. Hence hardware floats may coexist with MPFR floats, and even in MPFR floats, some may have 100 bits of mantissa and some may have 150 bits of mantissa. If operations mix different kinds of floats, the most precise kind of floats are coerced to the less precise kind of floats.

7.1.2 Representation by hardware floats

A real is represented by a floating number d, that is

$$d = 2^{\alpha} * (1 + m), \quad 0 < m < 1, -2^{10} < \alpha < 2^{10}$$

If $\alpha > 1 - 2^{10}$, then $m \geq 1/2$, and d is a normalized floating point number, otherwise d is denormalized ($\alpha = 1 - 2^{10}$). The special exponent 2^{10} is used to represent plus or minus infinity and NaN (Not a Number). A hardware float is made of 64 bits:

- the first bit is for the sign of d (0 for '+' and 1 for '-')
- the 11 following bits represents the exponent, more precisely if α denotes the integer from the 11 bits, the exponent is $\alpha + 2^{10} - 1$,
- the 52 last bits codes the mantissa m, more precisely if M denotes the integer from the 52 bits, then $m = 1/2 + M/2^{53}$ for normalized floats and $m = M/2^{53}$ for denormalized floats.

Examples of representations of the exponent:

- $\alpha = 0$ is coded by 011 1111 1111
- $\alpha = 1$ is coded by 100 0000 0000
- $\alpha = 4$ is coded by 100 0000 0011
- $\alpha = 5$ is coded by 100 0000 0100
- $\alpha = -1$ is coded by 011 1111 1110
- $\alpha = -4$ is coded by 011 1111 1011
- $\alpha = -5$ is coded by 011 1111 1010
- $\alpha = 2^{10}$ is coded by 111 1111 1111
- $\alpha = 2^{-10} - 1$ is coded by 000 0000 000

Remark: $2^{-52} = 0.2220446049250313e - 15$

7.1.3 Examples of representations of normalized floats

- 3.1 :
  We have :

$$\begin{aligned} 3.1 &= 2 * (1 + \frac{1}{2} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{10}} + ...) \\ &= 2 * (1 + \frac{1}{2} + \sum_{k=1}^{\infty} (\frac{1}{2^{4*k+1}} + \frac{1}{2^{4*k+2}})) \end{aligned}$$

  hence $\alpha = 1$ and $m = \frac{1}{2} + \sum_{k=1}^{\infty} (\frac{1}{2^{4*k+1}} + \frac{1}{2^{4*k+2}})$. Hence the hexadecimal and binary representation of 3.1 is:

```
40 (01000000), 8 (00001000), cc (11001100), cc (11001100),
cc (11001100), cc (11001100), cc (11001100), cd (11001101),
```

the last octet is 1101, the last bit is 1, because the following digit is 1 (upper rounding).

- 3. :
  We have $3 = 2*(1+1/2)$. Hence the hexadecimal and binary representation of 3 is:

```
40 (01000000), 8 (00001000), 0 (00000000), 0 (00000000),
0 (00000000), 0 (00000000), 0 (00000000), 0 (00000000)
```

7.1.4 Difference between the representation of (3.1-3) and of 0.1

- representation of 0.1 :
  We have :

$$0.1 = 2^{-4}*(1+\frac{1}{2}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^8}+\frac{1}{2^9}+...) = 2^{-4}*\sum_{k=0}^{\infty}(\frac{1}{2^{4*k}}+\frac{1}{2^{4*k+1}})$$

hence $\alpha = 1$ and $m = \frac{1}{2}+\sum_{k=1}^{\infty}(\frac{1}{2^{4*k}}+\frac{1}{2^{4*k+1}})$, therefore the representation of 0.1 is

```
3f (00111111), b9 (10111001), 99 (10011001), 99 (10011001),
99 (10011001), 99 (10011001), 99 (10011001), 9a (10011010),
```

the last octet is 1010, indeed the 2 last bits 01 became 10 because the following digit is 1 (upper rounding).

- representation of a:=3.1-3 :
  Computing a is done by adjusting exponents (here nothing to do), then subtract the mantissa, and adjust the exponent of the result to have a normalized float. The exponent is $\alpha = -4$ (that corresponds at $2*2^{-5}$) and the bits corresponding to the mantissa begin at $1/2 = 2*2^{-6}$: the bits of the mantissa are shifted to the left of 5 positions and we have :

```
3f (00111111), b9 (10111001), 99 (10011001), 99 (10011001),
99 (10011001), 99 (10011001), 99 (10011001), 9a (10100000),
```

Therefore $a > 0.1$ and $a - 0.1 = 1/2^{50} + 1/2^{51}$ (since 100000-11010=110)

Remark
This is the reason why

```
floor(1/(3.1-3))
```

returns `9` and not `10` when `Digits:=14`.

7.2 Approx. evaluation : `evalf` `approx` and `Digits`

`evalf` or `approx` evaluates to a numeric approximation (if possible).
Input :

```
evalf(sqrt(2))
```

Output, if in the `cas` configuration (`Cfg` menu) `Digits=7` (that is hardware floats are used, and 7 digits are displayed) :

```
1.414214
```

You can change the number of digits in a command line by assigning the variable `DIGITS` or `Digits`. Input :

```
DIGITS:=20

evalf(sqrt(2))
```

Output :

```
1.4142135623730950488
```

Input :

```
evalf(10^-5)
```

Output :

```
1e-05
```

Input :

```
evalf(10^15)
```

Output :

```
1e+15
```

Input :

```
evalf(sqrt(2))*10^-5
```

Output :

```
1.41421356237e-05
```

7.3 Numerical algorithms

7.3.1 Approximate solution of an equation : newton

newton takes as arguments : an expression ex, the variable name of this expression (by default x), and three values a (by default a=0), eps (by default eps=1e-8) and nbiter (by default nbiter=12).
newton(ex,x,a,eps,nbiter) computes an approximate solution x of the equation ex=0 using the Newton algorithm with starting point x=a. The maximum number of iterations is nbiter and the precision is eps.
Input :

```
newton(x^2-2,x,1)
```

Output :

```
1.41421356237
```

Input :

```
newton(x^2-2,x,-1)
```

Output :

```
-1.41421356237
```

Input :

```
newton(cos(x)-x,x,0)
```

Output :

```
0.739085133215
```

7.3.2 Approximate computation of the derivative number : nDeriv

nDeriv takes as arguments : an expression ex, the variable name of this expression (by default x), and h (by default h=0.001).
nDeriv(ex,x,h) computes an approximated value of the derivative of the expression ex at the point x and returns :

```
(f(x+h)-f(x+h))/2*h
```

Input :

```
nDeriv(x^2,x)
```

Output :

```
((x+0.001)^2-(x+-0.001)^2)*500.0
```

Input :

```
subst(nDeriv(x^2,x),x=1)
```

Output :

```
2
```

Input :

```
nDeriv(exp(x^ 2),x,0.00001)
```

Output :

```
(exp((x+1e-05)^2)-exp((x+-1e-05)^2))*50000
```

Input :

```
subst(exp(nDeriv(x^ 2),x,0.00001),x=1)
```

Output :

```
5.43656365783
```

which is an approximate value of `2e=5.43656365692`.

7.3.3 Approximate computation of integrals : `romberg nInt`

`romberg` or `nInt` takes as arguments : an expression `ex`, the variable name of this expression (by default `x`), and two real values `a,b`.
`romberg(ex,x,a,b)` or `nInt(ex,x,a,b)` computes an approximated value of the integral $\int_a^b ex\ dx$ using the Romberg method. The integrand must be sufficiently regular for the approximation to be accurate. Otherwise, `romberg` returns a list of real values, that comes from the application of the Romberg algorithm (the first list element is the trapezoid rule approximation, the next ones come from the application of the Euler-MacLaurin formula to remove successive even powers of the step of the trapezoid rule).
Input :

```
romberg(exp(x^2),x,0,1)
```

Output :

```
1.46265174591
```

7.3.4 Approximate solution of y'=f(t,y) : `odesolve`

- Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$.
  `odesolve(f(t,y),[t,y],[t0,y0],t1)` or
  `odesolve(f(t,y),t=t0..t1,y,y0)` or
  `odesolve(t0..t1,f,y0)` or
  `odesolve(t0..t1,(t,y)->f(t,y),y0)`
  returns an approximate value of $y(t1)$ where $y(t)$ is the solution of:

  $$y'(t) = f(t, y(t)), \quad y(t0) = y0$$

- `odesolve` accepts an optional argument for the discretization of `t` (`tstep=value`). This value is passed as initial tstep value to the numeric solver from the GSL (Gnu Scientific Library), it may be modified by the solver. It is also used to control the number of iterations of the solver by `2*(t1-t0)/tstep` (if the number of iterations exceeds this value, the solver will stops at a time $t < t1$).

- `odesolve` accepts `curve` as an optional argument. In that case, `odesolve` returns the list of all the $[t, [y(t)]]$ values that were computed.

Input :

```
odesolve(sin(t*y),[t,y],[0,1],2)
```

or :

```
odesolve(sin(t*y),t=0..2,y,1)
```

or :

```
odesolve(0..2,(t,y)->sin(t*y),1)
```

or define the function :

```
f(t,y):=sin(t*y)
```

and input :

```
odesolve(0..2,f,1)
```

Output :

```
[1.82241255675]
```

Input :

```
odesolve(0..2,f,1,tstep=0.3)
```

Output :

```
[1.82241255675]
```

Input :

```
odesolve(sin(t*y),t=0..2,y,1,tstep=0.5)
```

Output :

```
[1.82241255675]
```

Input :

```
odesolve(sin(t*y),t=0..2,y,1,tstep=0.5,curve)
```

Output :

```
[[0.760963063136,[1.30972370515]],[1.39334557388,[1.86417104853]]]
```

7.3.5 Approximate solution of the system v'=f(t,v) : `odesolve`

- If v is a vector of variables $[x1, .., xn]$ and if f is given by a vector of expressions `[e1,...,en]` depending on t and of $[x1, .., xn]$, if the initial value of v at `t0` is the vector $[x10, ..., xn0]$ then the instruction

  ```
  odesolve([e1,..,en],t=t0..t1,[x1,...,xn],
                  [x10,...,xn0])
  ```

  returns an approximated value of v at $t = t1$. With the optional argument `curve`, `odesolve` returns the list of the intermediate values of $[t, v(t)]$ computed by the solver.

  Example, to solve the system

  $$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \end{aligned}$$

  Input :

  ```
  odesolve([-y,x],t=0..pi,[x,y],[0,1])
  ```

  Output :

  ```
  [-1.79045146764e-15,-1]
  ```

- If f is a function from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$.
  `odesolve(t0..t1,(t,v)->f(t,v),v0)` or
  `odesolve(t0..t1,f,v0)`
  computes an approximate value of $v(t1)$ where the vector $v(t)$ in $\mathbb{R}^n$ is the solution of

  $$v'(t) = f(t, v(t)), v(t0) = v0$$

  With the optional argument `curve`, `odesolve` returns the list of the intermediate value $[t, v(t)]$ computed by the solver.

  Example, to solve the system :

  $$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \end{aligned}$$

  Input :

  ```
  odesolve(0..pi,(t,v)->[-v[1],v[0]],[0,1])
  ```

  Or define the function:

  ```
  f(t,v):=[-v[1],v[0]]
  ```

  then input :

```
odesolve(0..pi,f,[0,1])
```

Output :

```
[-1.79045146764e-15,-1]
```

Alternative input :

```
odesolve(0..pi/4,f,[0,1],curve)
```

Output :

```
[[0.1781,[-0.177159948386,0.984182072936]],
[0.3781,[-0.369155338156,0.929367707805]],
[0.5781,[-0.54643366953,0.837502384954]],
[0.7781,[-0.701927414872,0.712248484906]]]
```

7.4 Solve equations with fsolve nSolve

fsolve or nSolve solves numeric equations (unlike solve or proot, it is not limited to polynomial equations) of the form:

$$f(x) = 0, \quad x \in]a, b[$$

fsolve or nSolve accepts a last optional argument, the name of an iterative algorithm to be used by the GSL solver. The different methods are explained in the following section.

7.4.1 fsolve or nSolve with the option bisection_solver

This algorithm of dichotomy is the simplest but also generically the slowest. It encloses the zero of a function on an interval. Each iteration, cuts the interval into two parts. We compute the middle point value. The function sign at this point, gives us the half-interval on which the next iteration will be performed.
Input :

```
fsolve((cos(x))=x,x,-1..1,bisection_solver)
```

Output :

```
[0.739085078239,0.739085137844]
```

7.4.2 fsolve or nSolve with the option brent_solver

The Brent method interpolates of f at three points, finds the intersection of the interpolation with the x axis, computes the sign of f at this point and chooses the interval where the sign changes. It is generically faster than bisection.
Input :

```
fsolve((cos(x))=x,x,-1..1,brent_solver)
```

Output :

```
[0.73908513321 5,0.739085133215]
```

7.4.3 `fsolve` or `nSolve` with the option `falsepos_solver`

The "false position" algorithm is an iterative algorithm based on linear interpolation : we compute the value of f at the intersection of the line $(a, f(a))$, $(b, f(b))$ with the x axis. This value gives us the part of the interval containing the root, and on which a new iteration is performed.
The convergence is linear but generically faster than bisection.
Input :

```
fsolve((cos(x))=x,x,-1..1,falsepos_solver)
```

Output :

```
[0.739085133215,0.739085133215]
```

7.4.4 `fsolve` or `nSolve` with the option `newton_solver`

`newton_solver` is the standard Newton method. The algorithm starts at an initial value x_0, then we search the intersection x_1 of the tangent at x_0 to the graph of f, with the x axis, the next iteration is done with x_1 instead of x_0. The x_i sequence is defined by

$$x_0 = x_0, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the Newton method converges, it is a quadratic convergence for roots of multiplicity 1.
Input :

```
fsolve((cos(x))=x,x,0,newton_solver)
```

Output :

```
0.739085133215
```

7.4.5 `fsolve` or `nSolve` with the option `secant_solver`

The secant method is a simplified version of the Newton method. The computation of x_1 is done using the Newton method. The computation of $f'(x_n), n > 1$ is done approximately. This method is used when the computation of the derivative is expensive:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'_{est}}, \quad f'_{est} = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}$$

The convergence for roots of multiplicity 1 is of order $(1 + \sqrt{5})/2 \approx 1.62...$.
Input :

```
fsolve((cos(x))=x,x,-1..1,secant_solver)
```

Output :

```
[0.739085078239,0.739085137844]
```

Input :

```
fsolve((cos(x))=x,x,0,secant_solver)
```

Output :

```
0.739085133215
```

7.4.6 fsolve or nSolve with the option steffenson_solver

The Steffenson method is generically the fastest method.
It combines the Newton method with a "delta-two" Aitken acceleration : with the Newton method, we obtain the sequence x_i and the convergence acceleration gives the Steffenson sequence

$$R_i = x_i - \frac{(x_{i+1} - x_i)^2}{(x_{i+2} - 2x_{i+1} + x_i)}$$

Input :

```
fsolve(cos(x)=x,x,0,steffenson_solver)
```

Output :

```
0.739085133215
```

7.5 Solve systems with fsolve

Xcas provides six methods (inherited from the GSL) to solve numeric systems of equations of the form $f(x) = 0$:

- Three methods use the jacobian matrix $f'(x)$ and their names are terminated with j_solver.
- The three other methods use approximation for $f'(x)$ and use only f.

All methods use an iteration of Newton kind

$$x_{n+1} = x_n - f'(x_n)^{-1} * f(x_n)$$

The four methods hybrid*_solver use also a method of gradient descent when the Newton iteration would make a too large step. The length of the step is computed without scaling for hybrid_solver and hybridj_solver or with scaling (computed from $f'(x_n)$) for hybrids_solver and hybridsj_solver.

7.5.1 fsolve with the option dnewton_solver

Input :

```
fsolve([x^2+y-2,x+y^2-2],[x,y],[2,2],dnewton_solver)
```

Output :

```
[1.0,1.0]
```

7.5.2 fsolve with the option hybrid_solver

Input :

```
fsolve([x^2+y-2,x+y^2-2],[x,y],[2,2],
cos(x)=x,x,0,hybrid_solver)
```

Output :

```
[1.0,1.0]
```

7.5.3 `fsolve` with the option `hybrids_solver`

Input :

```
fsolve([x^2+y-2,x+y^2-2],[x,y],[2,2],hybrids_solver)
```

Output :

```
[1.0,1.0]
```

7.5.4 `fsolve` with the option `newtonj_solver`

Input :

```
fsolve([x^2+y-2,x+y^2-2],[x,y],[0,0],newtonj_solver)
```

Output :

```
[1.0,1.0]
```

7.5.5 `fsolve` with the option `hybridj_solver`

Input :

```
fsolve([x^2+y-2,x+y^2-2],[x,y],[2,2],hybridj_solver)
```

Output :

```
[1.0,1.0]
```

7.5.6 `fsolve` with the option `hybridsj_solver`

Input :

```
fsolve([x^2+y-2,x+y^2-2],[x,y],[2,2],hybridsj_solver)
```

Output :

```
[1.0,1.0]
```

7.6 Numeric roots of a polynomial : `proot`

`proot` takes as argument a squarefree polynomial, either in symbolic form or as a list of polynomial coefficients (written by decreasing order).
`proot` returns a list of the numeric roots of this polynomial.
To find the numeric roots of $P(x) = x^3 + 1$, input :

```
proot([1,0,0,1])
```

or :

```
proot(x^3+1)
```

Output :

```
[0.5+0.866025403784*i,0.5-0.866025403784*i,-1.0]
```

To find the numeric roots of $x^2 - 3$, input :

```
proot([1,0,-3])
```

or :

```
proot(x^2-3)
```

Output :

```
[1.73205080757,-1.73205080757]
```

7.7 Numeric factorization of a matrix : cholesky qr lu svd

Matrix numeric factorizations of

- Cholesky,
- QR,
- LU,
- svd,

are described in section 5.50.

Chapter 8

Unit objects and physical constants

The `Phys` menu contains:

- the physical constants (`Constant` sub-menu),
- the unit conversion functions (`Unit_convert` sub-menu),
- the unit prefixes (`Unit_prefix` sub-menu)
- the unit objects organized by subject

8.1 Unit objects

8.1.1 Notation of unit objects

A unit object has two parts : a real number and a unit expression (a single unit or a multiplicative combination of units). The two parts are linked by the character _ ("underscore"). For example `2_m` for 2 meters. For composite units, parenthesis must be used, e.g. `1_(m*s)`.
If a prefix is put before the unit then the unit is multiplied by a power of 10. For example `k` or `K` for kilo (indicate a multiplication by 10^3), `D` for deca (indicate a multiplication by 10), `d` for deci (indicate a multiplication by 10^{-1}) etc...
Input :

```
10.5_m
```

Output :

```
a unit object of value 10.5 meters
```

Input :

```
10.5_km
```

Output :

```
a unit object of value 10.5 kilometers
```

8.1.2 Computing with units

`Xcas` performs usual arithmetic operations (+, -, *, /, ^) on unit objects. Different units may be used, but they must be compatible for + and -. The result is an unit object

- for the multiplication and the division of two unit objects `_u1` and `_u2` the unit of the result is written `_(u1*u2)` or `_(u1/u2)`.
- for an addition or a subtraction of compatible unit objects, the result is expressed with the same unit as the first term of the operation.

Input :

```
1_m+100_cm
```

Output :

```
2_m
```

Input :

```
100_cm+1_m
```

Output :

```
200_cm
```

Input :

```
1_m*100_cm
```

Output :

```
1_m^2
```

8.1.3 Convert units into MKSA units : `mksa`

`mksa` converts a unit object into a unit object written with the compatible `MKSA` base unit.
Input :

```
mksa(15_C)
```

Output :

```
15_(s*A)
```

8.1.4 Convert units : `convert`

`convert` convert units : the first argument is an unit object and the second argument is the new unit (which must be compatible).
Input :

```
convert(1_h,_s)
```

Output :

```
3600_s
```

Input :

```
convert(3600_s,_h)
```

Output :

```
1_h
```

8.1.5 Factorize a unit : `ufactor`

`ufactor` factorizes a unit in a unit object : the first argument is a unit object and the second argument is the unit to factorize.
The result is an unit object multiplied by the remaining `MKSA` units.
Input :

```
ufactor(3_J,_W)
```

Output :

```
3_(W*s)
```

Input :

```
ufactor(3_W,_J)
```

Output :

```
3_(J/s)
```

8.1.6 Simplify a unit : `usimplify`

`usimplify` simplifies a unit in an unit object.
Input :

```
usimplify(3_(W*s))
```

Output :

```
3_J
```

8.1.7 Unit prefixes

You can insert a unit prefix in front of a unit to indicate a power of ten.
The following table gives the available prefixes:

Prefix	Name	(*10^) n	Prefix	Name	(*10^) n
Y	yota	24	d	deci	-1
Z	zeta	21	c	cent	-2
E	exa	18	m	mili	-3
P	peta	15	mu	micro	-6
T	tera	12	n	nano	-9
G	giga	9	p	pico	-12
M	mega	6	f	femto	-15
k or K	kilo	3	a	atto	-18
h or H	hecto	2	z	zepto	-21
D	deca	1	y	yocto	-24

Remark
You cannot use a prefix with a built-in unit if the result gives another built-in unit. For example, `1_a` is one are, but `1_Pa` is one pascal and not `10^15_a`.

8.2 Constants

8.2.1 Notation of physical constants

If you want to use a physical constants inside Xcas, put its name between two characters _ ("underscore"). Don't confuse physical constants with symbolic constants, for example, e, π are symbolic constants as `_c_`, `_NA_` are physical constants.
Input :

```
_c_
```

Output speed of light in vacuum :

```
299792458_m*s^-1
```

Input :

```
_NA_
```

Output Avogadro's number :

```
6.0221367e23_gmol^-1
```

8.2.2 Constants Library

The physical constants are in the `Phys` menu, `Constant` sub-menu. The following table gives the Constants Library :

Name	Description
_NA_	Avogadro's number
_k_	Boltzmann constant
_Vm_	Molar volume
_R_	Universal gas constant
StdT	Standard temperature
StdP	Standard pressure
sigma	Stefan-Boltzmann constant
_c_	Speed of light in vacuum
epsilon0	Permitivity of vacuum
_mu0_	Permeability of vacuum
_g_	Acceleration of gravity
_G_	Gravitational constant
_h_	Planck's constant
hbar	Dirac's constant
_q_	Electron charge
_me_	Electron rest mass
qme	q/me (Electron charge/mass)
_mp_	Proton rest mass
mpme	mp/me (proton mass/electron mass)
alpha	Fine structure constant
phi	Magnetic flux quantum
_F_	Faraday constant
Rinfinity	Rydberg constant
_a0_	Bohr radius
muB	Bohr magneton
muN	Nuclear magneton
lambda0	Photon wavelength (ch/e)
_f0_	Photon frequency (e/h)
lambdac	Compton wavelength
rad	1 radian
twopi	2*pi radians
angl	180 degrees angle
_c3_	Wien displacement constant
_kq_	k/q (Boltzmann/electron charge)
epsilon0q	epsilon0/q (permitivity /electron charge)
qepsilon0	q*epsilon0 (electron charge *permitivity)
epsilonsi	Silicium dielectric constant
epsilonox	Bioxyd of silicium dielectric constant
_I0_	Reference intensity

To have the value of a constant, input the constant name in the command line of `Xcas` and evaluate with `enter` (don't forget to put _ at the beginning and at the end of the constant name).