

Introduction to QPA

Part 2

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Third GAP Days

Outline

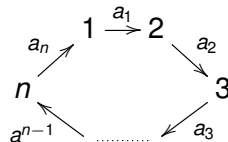
- 1 Basic functions
 - Special algebras
 - Modules
 - Homomorphisms

- 2 Chain complexes

Nakayama algebras

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} n$$

or



A Nakayama algebra

$$A = kQ / \langle \rho \rangle \quad Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \quad \rho = \{ab\}$$

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Indecomposable projective A -modules:

$$P1: k \xrightarrow{1} k \longrightarrow 0 \longrightarrow 0$$

$$P2: 0 \longrightarrow k \xrightarrow{1} k \xrightarrow{1} k$$

$$P3: 0 \longrightarrow 0 \longrightarrow k \xrightarrow{1} k$$

$$P4: 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow k$$

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Admissible sequence: $(2, 3, 2, 1)$

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Admissible sequence: $(2, 3, 2, 1)$

```
gap> NakayamaAlgebra([2,3,2,1], Rationals);
```


Truncated path algebras

- kQ/I , where I generated by all paths of length n

```
gap> Q := Quiver(3, [[1,2,"a"],  
                    [2,1,"b"],  
                    [2,2,"c"]]);;  
gap> A := TruncatedPathAlgebra(Rationals, Q, 3);
```

Recall: Modules (representations) in QPA

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$M: k \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);;  
gap> kQ := PathAlgebra(Rationals, Q);;  
gap> M := RightModuleOverPathAlgebra  
      (kQ, [1,2,1],  
        [[ "a", [[2,0]] ], [ "b", [[4],[-1]] ]]);  
<[ 1, 2, 1 ]>
```

Module attributes

$$M: k^1 \xrightarrow{(2 \ 0)} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k^1$$

- RightActingAlgebra: kQ
- LeftActingDomain: k
- DimensionVector: $(1, 2, 1)$
- MatricesOfPathAlgebraModule: $\left((2 \ 0), \begin{pmatrix} 4 \\ -1 \end{pmatrix} \right)$
- Dimension: $4 = 1 + 2 + 1$

Module attributes

$$M: k^1 \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k^2$$

- Basis:

$$1 \rightarrow (0, 0) \rightarrow 0$$

$$0 \rightarrow (1, 0) \rightarrow 0$$

$$0 \rightarrow (0, 1) \rightarrow 0$$

$$0 \rightarrow (0, 0) \rightarrow 1$$

- MinimalGeneratingSetOfModule:

$$1 \rightarrow (0, 0) \rightarrow 0$$

$$0 \rightarrow (0, 0) \rightarrow 1$$

Submodules

$$N \xhookrightarrow{i} M$$

- Categorical view of submodules
- A submodule is given by an inclusion homomorphism
- A submodule is not a subset

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- Categorical view of submodules
- A submodule is given by an inclusion homomorphism
- A submodule is not a subset
- SubRepresentation: N
- SubRepresentationInclusion: i

Direct sum

$$\begin{array}{ccc} M_1 & \xhookrightarrow{i_1} & \\ & & \\ M_2 & \xhookrightarrow{i_2} & M_1 \oplus M_2 \oplus M_3 \xrightarrow{p_2} M_2 \\ & & \\ M_3 & \xhookrightarrow{i_3} & \\ & & \\ & & M_2 \xrightarrow{p_1} \\ & & \\ & & M_3 \xrightarrow{p_3} \end{array}$$

- `DirectSumOfQPAModules`: $M_1 \oplus M_2 \oplus M_3$
- `DirectSumInclusions`: (i_1, i_2, i_3)
- `DirectSumProjections`: (p_1, p_2, p_3)

Radical, socle and top

$$\begin{array}{ccc} \text{rad } M & \xrightarrow{i} & M \\ & \uparrow j & \\ & \text{soc } M & \end{array} \quad M \xrightarrow{p} \text{top } M$$

- `RadicalOfModule`: $\text{rad } M$
- `RadicalOfModuleInclusion`: i
- `SocleOfModule`: $\text{soc } M$
- `SocleOfModuleInclusion`: j
- `TopOfModule`: $\text{top } M$
- `TopOfModuleInclusion`: p

Modules: equality and isomorphism

Three ways to compare modules M and N :

- `IsIdenticalObj (M, N)`
- $M = N$
- `IsomorphicModules (M, N)`

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For isomorphic modules:

- `IsomorphismOfModules (M, N)`
produces isomorphism $M \xrightarrow{\cong} N$

Simple modules

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Simple kQ -modules:

$$S_1: k \longrightarrow 0 \longrightarrow 0$$

$$S_2: 0 \longrightarrow k \longrightarrow 0$$

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In QPA: `SimpleModules` gives (S_1, S_2, S_3)

Indecomposable projective modules

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Indecomposable projective kQ -modules:

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In QPA: `IndecProjectiveModules` gives (P_1, P_2, P_3)

Indecomposable injective modules

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Indecomposable injective kQ -modules:

$$I_1: k \longrightarrow 0 \longrightarrow 0$$

$$I_2: k \xrightarrow{1} k \longrightarrow 0$$

$$I_3: k \xrightarrow{1} k \xrightarrow{1} k$$

Indecomposable injective modules

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In QPA: `IndecInjectiveModules` gives (I_1, I_2, I_3)

Homomorphisms

Recall:

$$\begin{array}{ccccc}
 M: & 0 & \longrightarrow & k & \xrightarrow{5} & k \\
 \downarrow h & \downarrow & & \downarrow (3 \ 2) & & \downarrow 1 \\
 N: & k & \xrightarrow{(0 \ 3)} & k^2 & \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} & k
 \end{array}$$

```
gap> h := RightModuleHomOverAlgebra
      (M, N, [ [[0]], [[3,2]], [[1]] ]);
<<[ 0, 1, 1 ]> ---> <[ 1, 2, 1 ]>>
```

Hom spaces

- `HomOverAlgebra (M, N)` gives k -basis for $\text{Hom}_A(M, N)$.
- k -structure on homomorphisms in $\text{Hom}_A(M, N)$:
use `f+g` and `scalar*f`

Composition of homomorphisms

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

Composition: $f \star g$

Kernel, Cokernel, Image

$$M \xrightarrow{f} N$$

Kernel, Cokernel, Image

$$\ker f \hookrightarrow M \xrightarrow{f} N$$

- Kernel: $\ker f$
- KernelInclusion: i

Kernel, Cokernel, Image

$$\ker f \hookrightarrow M \xrightarrow{f} N \xrightarrow{p} \text{coker } f$$

- Kernel: $\ker f$
- KernelInclusion: i
- CoKernel: $\text{coker } f$
- CoKernelProjection: p

Kernel, Cokernel, Image

$$\ker f \hookrightarrow M \xrightarrow{f} N \xrightarrow{p} \text{coker } f$$

$\begin{array}{c} \uparrow j \\ \text{im } f \end{array}$

- Kernel: $\ker f$
- KernelInclusion: i
- CoKernel: $\text{coker } f$
- CoKernelProjection: p
- Image: $\text{im } f$
- ImageInclusion: j

Chain complexes

$$C: \cdots \rightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \rightarrow \cdots$$

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- To represent a chain complex: Need infinite list $(\dots, d_2, d_1, d_0, d_{-1}, \dots)$ of differentials.

Chain complexes

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- To represent a chain complex: Need infinite list $(\dots, d_2, d_1, d_0, d_{-1}, \dots)$ of differentials.
- Can not store all the differentials.
- Need to describe them with finite data.

Chain complexes in QPA

Divide complex in three parts:

$$\underbrace{\dots \xrightarrow{d_{b+m+1}} \xrightarrow{d_{b+m}}}_{\text{"positive"} \atop (\text{infinite})} \underbrace{\xrightarrow{d_{b+m-1}} \dots \xrightarrow{d_b}}_{\text{"middle"} \atop (\text{finite})} \underbrace{\xrightarrow{d_{b-1}} \xrightarrow{d_{b_2}} \dots}_{\text{"negative"} \atop (\text{infinite})}$$

Chain complexes in QPA

Divide complex in three parts:

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 \end{array}$$

- Middle part: List of differentials
- Positive/negative part: Three possibilities

Possibilities for the infinite parts

Consider the positive part:

$$\dots \xrightarrow{d_3} \xrightarrow{d_2} \xrightarrow{d_1}$$

(assuming it starts with d_1)

Possibility 1: Repeating list

- The same list (r_1, \dots, r_n) of differentials repeated infinitely.

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$$\dots \xrightarrow{d_9} \xrightarrow{d_8} \xrightarrow{d_7} \xrightarrow{d_6} \xrightarrow{d_5} \xrightarrow{d_4} \xrightarrow{d_3} \xrightarrow{d_2} \xrightarrow{d_1}$$

$r_3 \quad r_2 \quad r_1$

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- Special case: Zero

Possibility 2: Inductive function

- Initial differential d_1
- Function f producing d_{i+1} from d_i

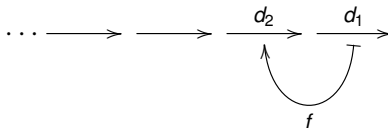
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$$\dots \longrightarrow \longrightarrow \longrightarrow \xrightarrow{d_1}$$

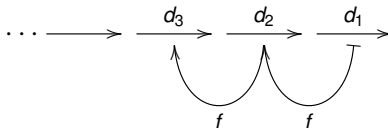
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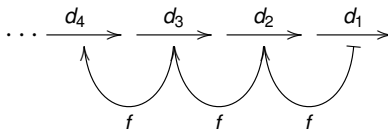
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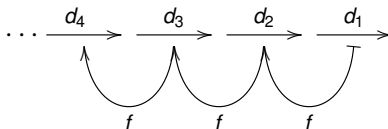
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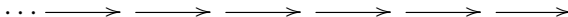
- Can convert to “repeating list” if repetition is detected

Possibility 3: Positional function

- Function f producing d_i from i .

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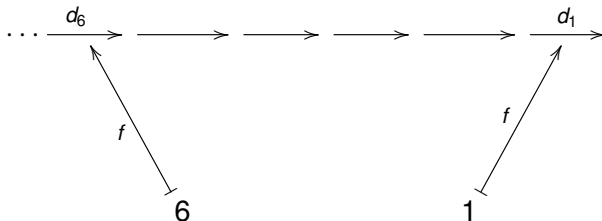
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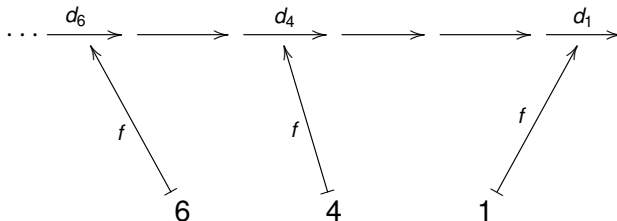
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Creating a chain complex

$$\underbrace{\dots \xrightarrow{d_{b+m+1}} \xrightarrow{d_{b+m}}}_{\text{positive}} \underbrace{\xrightarrow{d_{b+m-1}} \dots \xrightarrow{d_b}}_{\text{middle}} \underbrace{\xrightarrow{d_{b-1}} \xrightarrow{d_{b_2}} \dots}_{\text{negative}}$$

Must specify:

- Position b
- Middle part: (d_b, \dots, d_{b+m-1})
- Positive part: Repeating list or inductive function or positional function
- Negative part: Repeating list or inductive function or positional function

Special complex constructors

- `ZeroComplex`
- `FiniteComplex`
- `StalkComplex`

Projective resolutions

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

- ProjectiveResolution