

Introduction to QPA

Part 1

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Third GAP Days

Outline

- 1 About QPA
 - What is QPA?
 - Obtaining QPA
- 2 Basic structures
 - Quivers
 - Path algebras
 - Modules

What is QPA?

- QPA = Quivers and Path Algebras
- GAP package for computations with quotients of path algebras and their modules

Obtaining QPA

- QPA is distributed with GAP (from version 4.7.8)
- Can also clone git repository from
`https://github.com/gap-system/qpq`
to follow QPA development
- Loading QPA in GAP:
`gap> LoadPackage("qpq");`

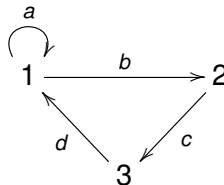
Basic structures

- Quivers
- Path algebras (modulo relations)
- Modules (representations)
and homomorphisms

Quivers

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

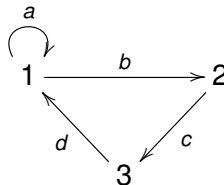
$$1 \begin{matrix} \xrightarrow{a} \\ \xrightarrow{b} \end{matrix} 2$$



Quivers

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$1 \begin{matrix} \xrightarrow{a} \\ \xrightarrow{b} \end{matrix} 2$$



Quiver: oriented graph (loops and multiple edges allowed)

Paths

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Paths in Q :

Paths

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Paths in Q :

- Length 0: e_1, e_2, e_3 (vertices/trivial paths)

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- Length 1: a, b (arrows)

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Paths in Q :

- Length 0: e_1, e_2, e_3 (vertices/trivial paths)
- Length 1: a, b (arrows)
- Length 2: ab (concatenation of a and b)

Paths

$$Q: 1 \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \end{array} 2$$

Paths in Q :

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$$Q: 1 \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \end{array} 2$$

Paths in Q :

- Length 0: e_1, e_2

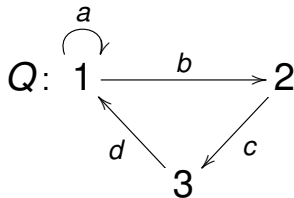
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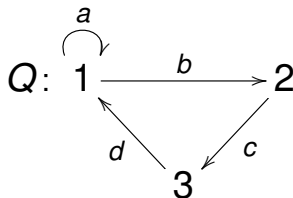
- Length 0: e_1, e_2
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Paths



Paths in Q :

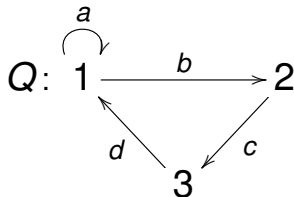
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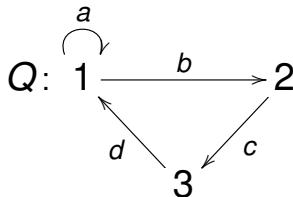
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Paths in Q :

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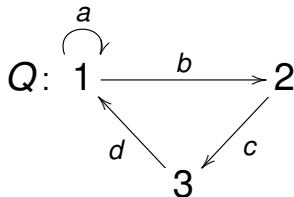
Paths



Paths in Q :

- Length 0: e_1, e_2, e_3
- Length 1: a, b, c, d
- Length 2: a^2, ab, bc, cd, da, db

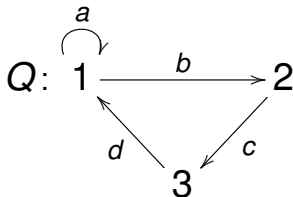
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Paths in Q :

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- Length 3: $a^3, a^2b, abc, bcd, cda, cdb, da^2, dab, dbc$

Paths



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- ...

Constructing a quiver in QPA

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);  
<quiver with 3 vertices and 2 arrows>
```

Path algebras

- Given quiver Q and field k
- Define *path algebra* kQ
- Basis: paths in Q
- Multiplication: concatenation of paths

Path algebras

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

k a field

Path algebras

$$\left. \begin{array}{l} Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \\ k \text{ a field} \end{array} \right\} \rightsquigarrow \text{path algebra } kQ$$

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- Multiplication:

$$e_1 \cdot e_1 = e_1$$

$$e_1 \cdot e_2 = 0$$

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$$a \cdot b = ab$$

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\cdot	e_1	e_2	e_3	a	b	ab
e_1	e_1	0	0	a	0	ab
e_2	0	e_2	0	0	b	0
e_3	0	0	e_3	0	0	0
a	0	a	0	0	ab	0
b	0	0	b	0	0	0
ab	0	0	ab	0	0	0

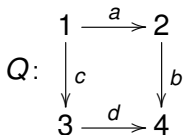
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<quiver with 3 vertices and 2 arrows>  
gap> kQ := PathAlgebra(Rationals, Q);  
<Rationals[<quiver with 3 vertices and 2 arrows>]>
```

Relations

- *Relation*: linear combination of paths with common source and target

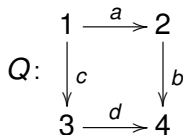


$$\sigma = \underbrace{ab - 2cd} \in kQ$$

source: 1
target: 4

Quotient of path algebra modulo relations

- Given path algebra kQ and a set $\rho \subseteq kQ$ of relations.
- Can create quotient algebra $kQ/\langle \rho \rangle$.



$$A = kQ/\langle ab - 2cd \rangle$$

Quotients in QPA

$$Q: \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ \downarrow c & & \downarrow b \\ 3 & \xrightarrow{d} & 4 \end{array} \quad A = kQ / \langle ab - 2cd \rangle$$

```
gap> Q := Quiver(4, [[1,2,"a"],[2,4,"b"],  
                    [1,3,"c"],[3,4,"d"]]);;  
gap> kQ := PathAlgebra(Rationals,Q);;  
gap> A := kQ/[kQ.a*kQ.b - 2*kQ.c*kQ.d];;  
gap> A.a * A.b;  
[(1)*a*b]  
gap> A.c * A.d;  
[(1/2)*a*b]
```

Admissible ideals

- $J \subseteq kQ$ ideal generated by the arrows
- Ideal $I \subseteq kQ$ *admissible* if $J^t \subseteq I \subseteq J^2$ for some $t \geq 2$
- If I admissible, then kQ/I finite-dimensional

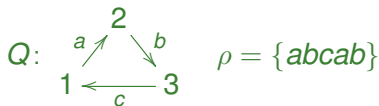
Admissible ideals – what does it mean?

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long paths must die

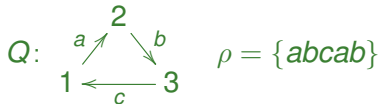


Admissible ideals – what does it mean?

$$\underline{J^t} \subseteq \underline{I} \subseteq \underline{J^2}$$

long paths must die

no single arrows in relations



Admissible ideals – why?

Algebras of the form kQ/I with I admissible ...

- ... are “almost all” finite-dimensional algebras, and
- ... have a nice theory.

Checking ideals for admissibility in QPA

```
gap> Q := Quiver(4, [[1,2,"a"],[2,4,"b"],  
                    [1,3,"c"],[3,4,"d"]]);;  
gap> kQ := PathAlgebra(Rationals,Q);;  
gap> I := Ideal(kQ, [kQ.a*kQ.b - 2*kQ.c*kQ.d]);;  
gap> IsAdmissibleIdeal(I);  
true
```

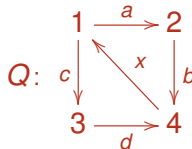
Algebras: summary

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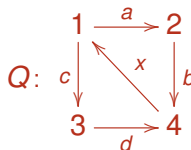
- a QUIVER Q ,



Algebras: summary

Take ...

- a QUIVER Q ,
- a FIELD k ,

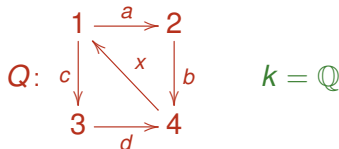


$$k = \mathbb{Q}$$

Algebras: summary

Take ...

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- a set $\rho \subseteq kQ$ of RELATIONS

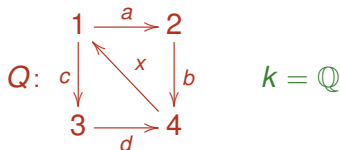


$$\rho = \{ab - cd, dxa\}$$

Algebras: summary

Take ...

- a **QUIVER** Q ,
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- a set $\rho \subseteq kQ$ of **RELATIONS** generating an **ADMISSIBLE** ideal



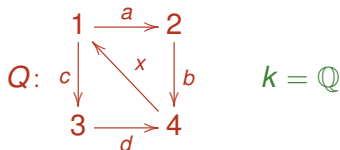
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$$J^t \subseteq \langle \rho \rangle \subseteq J^2$$

Algebras: summary

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$$\rho = \{ab - cd, dxa\}$$

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Let ... $A = kQ / \langle \rho \rangle$

Modules and representations

$$\text{mod } kQ \simeq \text{Rep}_k Q$$

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finitely generated kQ -modules

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representations of Q over k



Representations

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Want to make a representation R of Q .

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Given

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

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$$R: V_1 \longrightarrow V_2 \longrightarrow V_3$$

Start with the quiver, and put

- a vector space at each vertex,

Representations

Given

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Want to make a representation R of Q .

$$R: V_1 \xrightarrow{f_a} V_2 \xrightarrow{f_b} V_3$$

Start with the quiver, and put

- a vector space at each vertex,
- a linear transformation on each arrow.

A representation

$$Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$R: k \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

Creating a representation in QPA

$$R: k \xrightarrow{(2 \ 0)} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);;  
gap> kQ := PathAlgebra(Rationals, Q);;  
gap> M := RightModuleOverPathAlgebra  
      (kQ, [1,2,1],  
          [["a", [[2,0]]], ["b", [[4],[-1]]]]);  
<[ 1, 2, 1 ]>
```

Module structure of a representation

$$R: k \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

An element e of R :

$$e: 3 \longrightarrow \begin{pmatrix} 5 & 7 \end{pmatrix} \longrightarrow 4$$

Multiplying e with elements of kQ :

$$e \cdot v_1: 3 \longrightarrow \begin{pmatrix} 0 & 0 \end{pmatrix} \longrightarrow 0$$

$$e \cdot a: 0 \longrightarrow \begin{pmatrix} 6 & 0 \end{pmatrix} \longrightarrow 0$$

$$e \cdot b: 0 \longrightarrow \begin{pmatrix} 0 & 0 \end{pmatrix} \longrightarrow 13$$

Representations and relations

Given quiver with relations:

$$Q: \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ c \downarrow & & \downarrow b \\ 3 & \xrightarrow{d} & 4 \end{array} \quad \rho = \{ab - 2cd\}$$

A representation M of Q respects the relation $ab - 2cd$ if $f_a f_b - 2f_c f_d = 0$.

$$M: \begin{array}{ccc} V_1 & \xrightarrow{f_a} & V_2 \\ f_c \downarrow & & \downarrow f_b \\ V_3 & \xrightarrow{f_d} & V_4 \end{array}$$

Representations and relations

Representations respecting relation $ab - 2cd$?

$$\begin{array}{ccc}
 k & \xrightarrow{f_a=1} & k \\
 f_c=1 \downarrow & & \downarrow f_b=(1 \ 0) \\
 k & \xrightarrow{f_d=(0 \ 1)} & k^2
 \end{array}$$

Representations and relations

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 \quad \text{NO } (f_a f_b - 2 f_c f_d = (1 \ -1) \neq 0)$$

Representations and relations

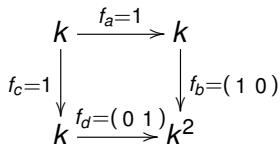
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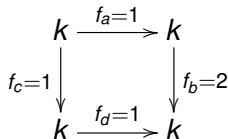
$$\begin{array}{ccc}
 k & \xrightarrow{f_a=1} & k \\
 f_c=1 \downarrow & & \downarrow f_b=2 \\
 k & \xrightarrow{f_d=1} & k
 \end{array}$$

Representations and relations

Representations respecting relation $ab - 2cd$?



NO ($f_a f_b - 2 f_c f_d = (1 \ -1) \neq 0$)



YES ($f_a f_b - 2 f_c f_d = 0$)

Representations and relations

For $A = kQ/\langle \rho \rangle$:

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finitely generated A -modules

representations of Q over k
respecting ρ

Module homomorphisms

For two modules M and N given as representations:

$$M: \quad V_1 \xrightarrow{f_a} V_2 \xrightarrow{f_b} V_3$$

$$N: \quad W_1 \xrightarrow{g_a} W_2 \xrightarrow{g_b} W_3$$

Module homomorphisms

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... a homomorphism $h: M \rightarrow N$ is given by:

Module homomorphisms

For two modules M and N given as representations:

$$\begin{array}{ccccc} M: & V_1 & \xrightarrow{f_a} & V_2 & \xrightarrow{f_b} & V_3 \\ & \downarrow h & & \downarrow h & & \downarrow h \\ N: & W_1 & \xrightarrow{g_a} & W_2 & \xrightarrow{g_b} & W_3 \end{array}$$

... a homomorphism $h: M \rightarrow N$ is given by:

- linear maps h_i for every vertex i ,

Module homomorphisms

For two modules M and N given as representations:

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 M: & V_1 & \xrightarrow{f_a} & V_2 & \xrightarrow{f_b} & V_3 \\
 & \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 \\
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 \end{array}$$

... a homomorphism $h: M \rightarrow N$ is given by:

- linear maps h_i for every vertex i ,
- commuting with the linear maps for the arrows.

Module homomorphisms in QPA

$$M: \quad 0 \longrightarrow k \xrightarrow{5} k$$

$$N: \quad k \xrightarrow{(0 \ 3)} k^2 \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"], [2,3,"b"]]);;
gap> kQ := PathAlgebra(Rationals, Q);;
gap> M := RightModuleOverPathAlgebra
      (kQ, [0,1,1], [{"b", [[5]]}]);;
gap> N := RightModuleOverPathAlgebra
      (kQ, [1,2,1], [{"a", [[0,3]]},
                     ["b", [[1],[1]]}]);;
```

Module homomorphisms in QPA

$$\begin{array}{ccccc}
 M: & 0 & \longrightarrow & k & \xrightarrow{5} & k \\
 & \downarrow & & \downarrow & & \downarrow \\
 & h & & (3 \ 2) & & (1) \\
 N: & k & \longrightarrow & k^2 & \longrightarrow & k \\
 & & & (0 \ 3) & & 1
 \end{array}$$

```

gap> h := RightModuleHomOverAlgebra
      (M, N, [ [[0]], [[3,2]], [[1]] ]);
<<[ 0, 1, 1 ]> ---> <[ 1, 2, 1 ]>>

```