

# Introduction to QPA

## Part 4

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# Outline

1 Examples of computations

2 Tilting modules

# $\Omega$ -periodic modules

$\Lambda$  – finite dimensional algebra

**Recall:** A module  $M$  is  $\Omega$ -periodic if  $\Omega_{\Lambda}^n(M) \simeq M$  for some positive integer  $n$ .

In some situations the period can indicate the degree of generators in the Hochschild cohomology ring,

$$\bigoplus_{i \geq 0} \text{Ext}_{\Lambda^{\text{env}}}^i(\Lambda, \Lambda).$$

# Periodic algebras

$\Lambda$  – finite dimensional  $k$ -algebra

$\Lambda^{\text{env}} = \Lambda^{\text{op}} \otimes_k \Lambda$  – enveloping algebra

**Recall:**  $\Lambda$  is a *periodic algebra* if  $\Lambda$  is a  $\Omega$ -periodic module, that is  $\Omega_{\Lambda^{\text{env}}}^n(\Lambda) \simeq \Lambda$  as  $\Lambda^{\text{env}}$ -modules.

## Facts:

- $\Lambda$  is a selfinjective algebra.
- All modules are  $\Omega$ -periodic.
- The Hochschild cohomology module nilpotent elements is isomorphic to  $k[x]$ , where the degree of  $x$  is the period.

# Finding quivers

**Recall:**  $\Lambda = kQ/I$  with  $I$  admissible is a basic algebra, that is,

$$\Lambda = \bigoplus_{i=1}^t P_i$$

with  $P_i$  indecomposable, then  $P_i \not\simeq P_j$  for  $i \neq j$ .

**Facts:**

- $\Lambda$  – finite dimensional algebra.
- $\text{rad } \Lambda = \langle \text{arrows} \rangle / I$ .
- $\Lambda / \text{rad } \Lambda \simeq \frac{kQ/I}{\langle \text{arrows} \rangle / I} \simeq \text{linear span of vertices}$ .
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$$\begin{aligned} \text{rad } \Lambda / \text{rad}^2 \Lambda &\simeq \frac{\langle \text{arrows} \rangle}{I} / \frac{\langle \text{arrows} \rangle^2}{I} \\ &\simeq \frac{\langle \text{arrows} \rangle}{\langle \text{arrows} \rangle^2} \\ &\simeq \text{linear span of arrows} \end{aligned}$$

# Finding quivers

$\Lambda$  – finite dimensional algebra with  $\Lambda / \text{rad } \Lambda \simeq k^n$  for some  $n$ .

Algorithm:

- ① Lift a complete set of orthogonal idempotents from  $\Lambda / \text{rad } \Lambda$  to a complete set of orthogonal idempotents in  $\Lambda$ , say  $\{e_i\}_{i=1}^n$  – the vertices.
- ② Compute  $e_i \text{rad } \Lambda / \text{rad}^2 \Lambda e_j$ , find a basis and lift back to  $e_i \text{rad } \Lambda e_j$  – the arrows from vertex  $i$  to vertex  $j$ .
- ③ Construct a quiver  $Q$  from this and a homomorphism  $\varphi: kQ \rightarrow \Lambda$ .
- ④ Find the kernel of  $\varphi$ .

# Trivial extensions

$\Lambda$  – finite dimensional algebra

$T(\Lambda) = \Lambda \oplus D(\Lambda)$  – *trivial extension*,

$$(\lambda, f) \cdot (\lambda', f') = (\lambda\lambda', \lambda f' + f\lambda)$$

- $\text{rad } T(\Lambda) = \text{rad } \Lambda \oplus D(\Lambda)$ .
- $\text{rad}^2 T(\Lambda) = \text{rad}^2 \Lambda \oplus D(\Lambda) \text{rad } \Lambda + \text{rad } \Lambda D(\Lambda)$ .
- $\frac{\text{rad } T(\Lambda)}{\text{rad}^2 T(\Lambda)} \simeq \frac{\text{rad } \Lambda}{\text{rad}^2 \Lambda} \oplus \frac{D(\Lambda)}{D(\Lambda) \text{rad } \Lambda + \text{rad } \Lambda D(\Lambda)}$
- $T(\Lambda)$  is a symmetric algebra,  $\Gamma \simeq D(\Gamma)$  as bimodules.

# AR-theory

Recall that a short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is *almost split exact* if it is not split exact and

- (i) for any not splittable epimorphism  $t: X \rightarrow C$  there is a homomorphism  $t': X \rightarrow B$  such that  $gt' = t$ ,
- (ii) for any not splittable monomorphism  $s: A \rightarrow Y$  there is a homomorphism  $s': A \rightarrow Y$  such that  $s'f = s$ .



# AR-theory

Facts:

- $C$  and  $A$  are indecomposable modules.
- $A \simeq D \operatorname{Tr} C$  and  $C \simeq \operatorname{Tr} D(A)$ .
- For any indecomposable non-projective module  $C$  and for any indecomposable non-injective module  $A$ , there is an almost split sequence ending in  $C$  and starting in  $A$ .
- An almost split sequence is a generator of the socle of  $\operatorname{Ext}_{\Lambda}^1(C, D \operatorname{Tr}(C))$  as an  $\operatorname{End}_{\Lambda}(C)$ -module.

# APR-tilting

$\Lambda = kQ$  – hereditary,  $Q$  no oriented cycle and connected.

$S$  simple projective module (and not injective)

$\Lambda$  is a classical tilting module  $T$ :

- $\text{pd}_{\Lambda} T \geq 1$ ,
- $\text{Ext}_{\Lambda}^1(T, T) = (0)$ ,
- the number of indecomposable non-isomorphic summands in  $T$  is equal to the number of isomorphism class of simple modules of  $\Lambda$ .

$\Lambda = P \oplus S \longrightarrow T = P \oplus \text{Tr } D(S)$  – APR-tilting